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COUNTING THE NUMBER OF EQUIVALENCE CLASSES OF BOREL AND COANALYTIC EQUIVALENCE RELATIONS

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It is the object of this paper to prove, within ordinary set theory (ZF):

If E is a coanalytic equivalence relation on the space of all real numbers and E has uncountably many equivalence classes, then there is a perfect set of mutually E inequivalent reals (hence E has $2^{\aleph_0}$ equivalence classes).

Since any Borel equivalence relation is coanalytic, our theorem in particular applies to Borel equivalence relations[1], and was previously unknown for them. Our theorem extends the well-known classical result (see [8]): every uncountable analytic set of reals includes a perfect set.

It should be stressed at once that "E is an equivalence class on the space of all reals" means, in particular, that for any real α, $\alpha E \alpha$, i.e. the field of E is precisely the set of all reals. Throughout this paper, ω denotes the set of natural numbers with the discrete topology. Thus ${}^{\omega}\omega = \{\alpha \mid \alpha : \omega \to \omega\}$, with the product topology, is Baire space, which is homeomorphic to the space of irrationals. If χ is a topological space, $S \subseteq \chi$ is *analytic* iff it is the projection on χ of a closed subset of $\chi \times {}^{\omega}\omega$. A set is *coanalytic* iff its complement is analytic. A coanalytic equivalence relation on χ is, of course, an equivalence relation on χ which, viewed as a subset of $\chi \times \chi$, is a coanalytic subset of $\chi \times \chi$ (and likewise for "Borel"). For reasons of conveneience, we shall in fact prove the variant of the above theorem obtained by replacing "the space of all reals" by "Baire space". This is obviously an equivalent form, since Baire space is homeomorphic to the space of irrationals: More information about most of these basic notions may be found in books by Kuratowski [5], Moschovakis [8], and in Section 1 of this paper.

It should also be mentioned that Burgess [2] has obtained, as a corollary of our theorem: If E is an analytic equivalence relation whose field is a set of reals and E has more than $\aleph_1$ equivalence classes, then there is a perfect set of mutually E inequivalent reals.

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[1]In the case of a Borel equivalence relations E, one can drop the requirement that the field be the whole set of reals since $\alpha E' \beta \leftrightarrow (\alpha E \beta) \vee (\alpha, \beta$ are outside of field $E)$ gives a coanalytic equivalence relation with one more equivalence class.

Since our theorem can be formulated in the language of so-called "analysis" or "second-order number theory", it might be expected that it could be proved within the usual axiomatic system for second-order number theory. Such is the case for almost all other statements of second-order number theory which are known to be provable in ordinary set theory and do not have a metamathematical content. In fact, the only statements known to be counterexamples to this rule relate to Borel determinacy. For example, the statement "all Borel games on $\aleph_0$ are determined" is a statement of second-order number theory which, by Martin [6], is a theorem of ZF but which, by Friedman [3], is not a theorem of second-order number theory. On the other hand, for all we know, it may yet be possible to find a proof of the theorem of this paper within second-order number theory.[2] In any case, the set-theoretical assumptions which we do employ are not in the least controversial.

Let Z be Zermelo set theory. Thus Z has among its axioms the so-called axiom schema of subsets, or Aussonderungsaxiom, but not the axiom schema of replacement. Our theorem can be proved in the theory $Z+$ there exist uncountably many cardinals. Note that Martin also uses $\aleph_1$ cardinals in his proof of Borel determinancy.

The questions which this theorem answers was, so far as I know, first raised by Friedman and included in an early, unpublished version of Friedman [4]. It was brought to my attention by Prikry, who obtained a solution for a very low level of the Borel hierarchy. Thus the question was open even for Borel equivalence relations.

1. The basic notions and a statement of the theorem

Let X be a set which, in this section, we understand to be endowed with the discrete topology. (In the paper $X = \omega$, $X = 2$, or $X =$ some ordinal, will be the chief examples.) ${}^{\omega}X$ denotes the set of functions from ω into X, and, if n is a natural number, since we make the identification $n = \{0, 1, \ldots, n-1\}$,

$$ {}^{n}X = \{s \mid s : n \to X\}. $$

If $s : n \to X$, we also write $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$. Lower case letters from the early part of the Greek alphabet will denote members of ${}^{\omega}X$, lower case letters from the latter part of the Roman alphabet denote finite sequences (i.e. members of ${}^{n}X$ for some $n \in \omega$ and some X) except for those few instances where x, y denote elements of an arbitrary topological space. Ordinals will be denoted by letters like η, ξ, μ, λ, ν, σ, τ.

[2] Since these lines were written, Leo Harrington [13] has in fact proved the theorem in second-order number theory, using a much simpler proof than the one given here.

We put ${}^{<\omega}X = \bigcup_{n\in\omega} {}^{n}X$, i.e. ${}^{<\omega}X$ is the set of all finite sequences of elements of X. If $\alpha \in {}^{\omega}X$, $n \in \omega$, then $\bar{\alpha}(n)$ is the restriction of α to $n = \{0, \ldots, n-1\}$, i.e.

$$\bar{\alpha}(n) = \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle.$$

If $s \in {}^{m}X$ and $n \leqslant m$, we write similarly

$$\bar{s}(n) = \langle s(0), s(1), \ldots, s(n-1)\rangle.$$

If $s, t \in {}^{<\omega}X$, we say that s *extends* t (also written $s \supseteq t$ or $t \subseteq s$) if and only if the length of s is at least as great as t and there is some n such that

$$t = \langle s(0), s(1), \ldots, s(n-1)\rangle$$

Similarly, if $\alpha \in {}^{\omega}X$, $s \in {}^{<\omega}X$, α extends s (also $s \subseteq \alpha$) iff there is some n such that $s = \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$. If $s \in {}^{<\omega}X$, we sha'l sometimes write

$$[s] = \{\alpha \in {}^{\omega}X \mid s \subseteq \alpha\}.$$

(It should always be clear what X is, so omission of reference to X should cause no confusion.) Similarly, if $s \in {}^{<\omega}X$, $t \in {}^{<\omega}Y$, then

$$[s, t] = \{\langle \alpha, \beta\rangle \in {}^{\omega}X \times {}^{\omega}Y \mid s \subseteq \alpha, t \subseteq \omega\}.$$

Recall that X is discrete and impose the product topology on ${}^{\omega}X$. Do the same for Y, Z and ${}^{\omega}Y$, ${}^{\omega}Z$. Clearly, if $U \subseteq {}^{\omega}X$, then U is open iff there is $A \subseteq {}^{<\omega}X$ such that $U =$ the set of extensions of sequences in A, i.e.

$$(\forall \alpha \in {}^{\omega}X)(\alpha \in U \leftrightarrow (\exists n \in \omega)\, \bar{\alpha}(n) \in A).$$

Similarly, if $U \subseteq {}^{\omega}X \times {}^{\omega}Y \times {}^{\omega}Z$, then U is open iff there is a ternary relation

$$R^1 \subseteq {}^{<\omega}X \times {}^{<\omega}Y \times {}^{<\omega}Z$$

such that

$$((\forall \langle \alpha, \beta, \gamma\rangle \in {}^{\omega}X \times {}^{\omega}Y \times {}^{\omega}Z)(\langle \alpha, \beta, \gamma\rangle \in U \leftrightarrow (\exists n \in \omega)\, R^1(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)))$$

where we have written $R^1(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n))$ instead of $\langle \bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)\rangle \in R^1$. Passing to complements, we see that a subset C of this space is closed iff there is a relation R such that

$$\langle \alpha, \beta, \gamma\rangle \in C \leftrightarrow (\forall n \in \omega)\, R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)).$$

A subset Q of a topological space χ is *analytic* iff it is the projection on χ of a closed subset of $\chi \times {}^{\omega}\omega$. Consider the special case $\chi = {}^{\omega}\omega \times {}^{\omega}\omega$. Thus $Q \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ is analytic iff there is a closed subset $C \subseteq {}^{\omega}\omega \times {}^{\omega}\omega \times {}^{\omega}\omega$ such that

$$\langle \alpha, \beta\rangle \in Q \leftrightarrow (\exists \gamma \in {}^{\omega}\omega) \quad \langle \alpha, \beta, \gamma\rangle \in C.$$

Using the above representation of C, we can conclude that there exists a ternary relation R on ${}^{<\omega}\omega$ such that

$$\langle \alpha, \beta\rangle \in Q \leftrightarrow (\exists \gamma \in {}^{\omega}\omega)(\forall n \in \omega)\, R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)).$$

It is this well-known representation of an analytic set on which we shall rely so heavily in the rest of the paper.

A subset T of ${}^{<\omega}\omega$ is a *tree* iff it is closed under subsequence, i.e. whenever $t \in T$ and $s \subseteq t$, $s \in {}^{<\omega}\omega$, then $s \in T$. If $\alpha \in {}^{\omega}\omega$, then α is a *path* through T iff, for all $n \in \omega$, $\bar{\alpha}(n) \in T$. Using the discussion of closed sets above, it is easy to see that a subset C of ${}^{\omega}\omega$ is closed iff there is a tree T such that C is the set of paths through T. Next, a tree T is said to be *perfect* iff T branches above every node of T, i.e. given $s \in T$, there are sequences $s_1 \supseteq s$, $s_2 \supseteq s$, $s_1 \in T$, $s_2 \in T$, $s_1 \neq s_2$, such that length s_1 = length s_2. Finally, P is a *perfect subset* of ${}^{\omega}\omega$ iff it is the set of paths through some nonempty perfect tree. This is easily seen to agree, for ${}^{\omega}\omega$, with the standard definition of (nonempty) perfect set: a nonempty closed set which has no isolated points. It is easy to show that, if $F : {}^{\omega}2 \to {}^{\omega}\omega$ is a 1–1 continuous function, then the range of F is a perfect set. In fact, every perfect set includes the range of such a function.

Theorem. *If $E \subseteq {}^{\omega}\omega \times {}^{\omega}\omega$ is a coanalytic equivalence relation such that $(\forall \alpha \in {}^{\omega}\omega)(\alpha E \alpha)$, and E has uncountably many equivalence classes, then there is a perfect set $P \subseteq {}^{\omega}\omega$ such that*

$$\alpha, \beta \in P, \quad \alpha \neq \beta \to \alpha E \beta \quad \textit{fails}.$$

Note first that the theorem becomes false if we replace "coanalytic" by "analytic". If $R \subseteq \omega \times \omega$ and $\alpha \in {}^{\omega}\omega$, say that α codes R iff

$$(\forall i, j \in \omega)(\alpha(2^i 3^j) = 0 \leftrightarrow \langle i, j\rangle \in R).$$

Define an equivalence relation E on ${}^{\omega}\omega$ by:

$\alpha E \beta \leftrightarrow [\alpha, \beta$ both fail to code linear orderings]
or [α, β both code linear orderings which are not well-orderings]
or [α, β code well-orderings having the same order type].

One can show that E is analytic most easily by using the fact that any relation on ${}^{\omega}\omega$ which is Σ^1_1 definable is analytic (see Addison [1] and Moschovakis [8]). The following definition of E can be formalized in a Σ^1_1 manner:

$\alpha E \beta \leftrightarrow [\alpha, \beta$ both fail to code linear orderings]
or $\exists \gamma, \delta\, [\alpha, \beta$ code linear orderings and γ, δ are infinite descending chains in the relations coded by α, β, resp.]
or $\exists \gamma\, [\gamma$ is an isomorphism between the relation coded by α and the relation coded by β].

The E equivalence classes are precisely the sets X, Y, and S_η, $\eta < \omega_1$, where $X = \{\alpha \mid \alpha$ fails to code linear ordering$\}$, $Y = \{\alpha \mid \alpha$ codes a linear ordering which is not a well-ordering$\}$ and $S_\eta = \{\alpha \mid \alpha$ codes a well-ordering of type $\eta\}$.

However, there cannot be a perfect set P of mutually E inequivalent elements. If such a P existed, we could assume that it contains only codes of well-orderings (since the result of removing finitely many elements, in this case two, from a given perfect set still includes a perfect set). Any two members of P would be codes of nonisomorphic well-orderings. This is well-known to be impossible (see Moschovakis [8]). The usual argument to show its impossibility is this: Let A be a coanalytic set which is not analytic. Since A is coanalytic, there is a continuous function F such that

$$(\forall \alpha \in {}^{\omega}\omega)(\alpha \in A \leftrightarrow F(\alpha) \text{ codes a well-ordering}).$$

(Here we have just used the fact that the set of codes of well-orderings is a "complete" coanalytic set—see Moschovakis [8].) Since the ordinals coded by members of P are cofinal in ω_1, we would have:

$$\alpha \in A \leftrightarrow (\exists \beta, \gamma)(\gamma \in P \text{ and } \beta \text{ is an order-preserving map of the relation coded by } F(\alpha) \text{ into the relation coded by } \gamma).$$

Thus A would by Σ^1_1 definable in terms of parameters (the "codes" for P and F) and would therefore be analytic, contrary to hypotheses. However, as was remarked at the beginning of this paper (at the end of the paragraph following the statement of the theorem), Burgess [2] has shown that if we simultaneously replace "coanalytic" by "analytic" and require that there by $>\aleph_1$ equivalence classes, the same conclusion can be obtained.

Finally, the example just discussed can easily be modified to show that we need the assumption $(\forall \alpha)(\alpha E \alpha)$. Just restrict the E used in the preceding two paragraphs to the set of codes of well-ordering and show that the resulting relation has a Π^1_1 definition.

We now discuss another tempting improvement of the theorem, which runs as follows: If S is an analytic relation on ${}^{\omega}\omega$ and there exists an uncountable set $X \subseteq {}^{\omega}\omega$ such that

$$(\forall \alpha, \beta \in X)(\alpha \neq \beta \rightarrow \alpha S \beta)$$

then there exists a set P of cardinality $2^{\aleph_0}$ with the same property. (The theorem would then be an immediate consequence, taking S to be the complement of E.) D.A. Martin found a (Borel) counterexample to this statement, which we reproduce here. (For cogniscenti, he puts $\alpha S \beta \leftrightarrow$ the Turing degrees of α and β are comparable.)

In effect, he notes that there is an Borel relation $<$ on ${}^{\omega}\omega$ having these properties:

(1) $\alpha < \alpha$, and $\alpha < \beta < \gamma \rightarrow \alpha < \gamma$ for all $\alpha, \beta, \gamma \in {}^{\omega}\omega$.

(2) for all $\beta \in {}^{\omega}\omega$, $\{\alpha \in {}^{\omega}\omega \mid \alpha < \beta\}$ is countable.

(3) if C is a countable subset of ${}^{\omega}\omega$, then there is $\beta \in {}^{\omega}\omega$ such that $C \subseteq \{\alpha \mid \alpha < \beta\}$. Then let $\alpha S \beta \leftrightarrow (\alpha < \beta$ or $\beta < \alpha)$. S satisfies the above hypothesis, because we can, by induction, form a sequence $\langle \alpha_\eta \mid \eta < \omega_1 \rangle$ of distinct elements

of ${}^{\omega}\omega$ such that $\eta < \eta' \to \alpha_\eta < \alpha_{\eta'}$. (To define α_η, first use (2) to get β such that $(\forall \eta' < \eta)(\beta S \alpha_{\eta'}$ fails) and then use (3) to obtain α_η such that $\{\beta\} \cup \{\alpha_{\eta'} < \eta\} \subseteq \{\gamma \mid \delta < \alpha_\eta\}$.) Clearly

$$X = \{\alpha_\eta \mid \eta < \omega_1\}$$

satisfies the condition $(\forall \alpha, \beta \in X)(\alpha \neq \beta \to \alpha S \beta)$. No set P of cardinality $> \aleph_1$ can be found with the property $(\forall \alpha, \beta \in P)(\alpha \neq \beta \to \alpha S \beta)$. To see this, put $\alpha \equiv \beta \leftrightarrow \alpha < \beta$ and $\beta < \alpha$. If P is such a set with cardinality $> \aleph_1$, look at $P/\equiv$. Using (1) and (2), it can be viewed as a linear ordering of cardinality $> \aleph_1$ in which each element has only countably many predecessors. No such linear ordering exists.

Finally, there are several ways of obtaining the desired $<$. Martin put: $\alpha < \beta \leftrightarrow \alpha$ is recursive in β. One can also take: $\alpha < \beta \leftrightarrow \alpha$ is first-order definable in $\langle \omega, +, \cdot, \beta \rangle$. To avoid metamathematics, one can proceed as follows: Put

$$\alpha = (\gamma)_i \quad \text{iff } (\forall n \in \omega)(\alpha(n) = \gamma(p_i^n))$$

where p_i is the ith prime. Say that $\alpha G \beta$ iff $(\exists i)(\alpha = (\beta)_i)$. Then put

$$\alpha < \beta \leftrightarrow \alpha = \alpha \quad \text{or} \quad (\exists \gamma_0, \gamma_1, \ldots, \gamma_n)(\gamma_0 = \alpha, \gamma_n = \beta, \text{ and } \gamma_0 G \gamma_1 G \gamma_2 \cdots G \gamma_n).$$

In all cases, one can see that $<$ is Δ_1^1 definable and hence Borel.

Thus the conjecture under discussion is false. However, I don't know a counterexample to this statement with its hypothesis strengthened to: card $X = \aleph_2$. Our example also shows that it is not sufficient to assume: S is symmetric and analytic, and, for every countable $C \subseteq {}^{\omega}\omega$, $\exists \alpha \in {}^{\omega}\omega - C$ such that $(\forall \beta \in C)(\alpha S \beta)$.

2. More preliminaries: definition of $\Vdash$, etc.

Let $\mathscr{P}$ be the set of all partial functions on ω, i.e. $f \in \mathscr{P}$ if and only if there is a subset X of ω such that $f : X \to \omega$. There is a natural way of topologizing $\mathscr{P}$: basic open sets are sets of the form $[p]$, where p is a member of $\mathscr{P}$ having finite domain and

$$[p] = \{f \in \mathscr{P} \mid f \text{ extends } p\}.$$

To say that f extends p means that domain f includes domain p, and, for all $m \in$ domain p, $f(m) = p(m)$.

Let $\mathscr{Y}$ be a topological space, and assume that $\mathscr{B}$ is the standard system of basic open neighborhoods for $\mathscr{Y}$. (Later, we shall say what this is in each relevant case.) If $F : \mathscr{Y} \to \mathscr{P}$, then, as can easily be seen, F is continuous if and only if, whenever $m, n \in \omega$ and $y \in \mathscr{Y}$ and $F(y)(m) = n$, then there is a set $U \in \mathscr{B}$ containing y such that $(\forall z \in U)(F(z)(m) = n)$. (When we write $F(z)(m) = n$, it is of course implicitly understood that $F(z)$ is defined at m.)

To clarify the notion of continuity further (and for later use) we introduce the definition of $\Vdash_L$, where $F:\mathcal{Y}\to\mathcal{P}$, $m, n\in\omega$.

Definition 2.1. $U\Vdash_L F(y)(m)=n$ if and only if $U\in\mathcal{B}$ and, for all $z\in U$, $F(z)(m)=n$. For short, we write $U\Vdash F(m)=n$ instead of $U\Vdash_L F(y)(m)=n$. Also, if $s\in{}^n\omega$, $U\Vdash \bar{F}(n)=s$ iff, for all $z\in U$, $\bar{F}(z)(n)=s$. (Clearly this is equivalent to saying: for all $i<n$, $U\Vdash F(i)=s(i)$.)

Thus $F:\mathcal{Y}\to\mathcal{P}$ is continuous if and only if:

$$(\forall m, n\in\omega)(\forall z\in\mathcal{Y})(F(z)(m)=n \text{ iff } (\exists U\in\mathcal{B})(z\in U \text{ and } U\Vdash F(m)=n)).$$

Definition 2.2. $F:\mathcal{Y}\to\mathcal{P}$ is *Cohen continuous* if and only if F is continuous and, for all $n\in\omega$,

$$\{y\in\mathcal{Y}\mid F(y) \text{ is defined at } n\} \text{ is dense in } \mathcal{Y}.$$

There is a characterization of Cohen continuity in terms of $\Vdash_L$: F is Cohen continuous iff F is continuuous and, for all $m\in\omega$ and $U\in\mathcal{B}$, there are $n\in\omega$ and $V\in\mathcal{B}$, $V\subseteq U$ such that $V\Vdash F(m)=n$.

Suppose $F:\mathcal{Y}\to\mathcal{P}$ is Cohen continuous. It is easy to show; if $U\in\mathcal{B}$:

$$U\Vdash F(m)=n \quad \text{iff } (\forall V\in\mathcal{B})(V\subseteq U)\to(\forall k\neq n)$$
$$\text{(it is not the case that } V\Vdash F(m)=k).$$

If $\mathcal{Y}$ is a space of the form ${}^\omega X$, where X is viewed as having the discrete topology and ${}^\omega X$ the product topology so induced (recall that ${}^\omega X=\{\alpha\mid\alpha:\omega\to X\}$), then we take the standard system of basic open neighborhoods to consist of all sets

$$[s]=\{\alpha\in{}^\omega X\mid\alpha \text{ extends } s\}$$

where s ranges over ${}^{<\omega}X$. Similarly, if $\mathcal{Y}$ is a subspace of ${}^\omega X$, the basic open neighborhoods are all nonempty sets of the form $[s]\cap\mathcal{Y}$. We shall write

$$s\Vdash F(m)=n$$

in place of

$$[s]\cap\mathcal{Y}\Vdash F(m)=n,$$

and analogously for $s\Vdash\bar{F}(m)=t$. Viewed in this light, the motivation for the term "Cohen continuous" becomes clear: a Cohen continuous function F from ${}^\omega X$ to $\mathcal{P}$ is essentially a forcing term with respect to the partial ordering ${}^{<\omega}X$ and $\Vdash_L$ is the forcing relation. Now suppose that $\mathcal{Y}$ is subspace of ${}^\omega X\times{}^\omega Y$. A typical basic open neighborhood of this space shall have the form $[s,t]\cap\mathcal{Y}$, where $s\in{}^{<\omega}X$, $t\in{}^{<\omega}Y$ and $[s,t]=\{\langle\alpha,\beta\rangle\in{}^\omega X\times{}^\omega Y\mid s\subseteq\alpha, t\subseteq\beta\}$. If $F:\mathcal{Y}\to\mathcal{P}$ is continuous, write $\langle s,t\rangle\Vdash F(m)=n$ iff $[s,t]\cap\mathcal{Y}\Vdash F(m)=n$. Also, where no confusion results, we shall write $[s,t]$ for $[s,t]\cap\mathcal{Y}$.

Let us return to the main theorem. We are assuming that E is a coanalytic equivalence relation. Suppose that, for all $\alpha, \beta \in {}^{\omega}\omega$,

$$\neg(\alpha E \beta) \leftrightarrow (\exists \gamma \in {}^{\omega}\omega)(\forall n \in \omega)\ R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)).$$

(See Section 1 and recall that $\bar{\alpha}(n) = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$.) From the assumption that E has uncountably many equivalence classes but no perfect set of mutually E inequivalent elements, we wish to derive a contradiction. In fact, we shall obtain the contradiction by obtaining continuous functions

$$H : {}^{\omega}2 \to {}^{\omega}\omega, \qquad K : \{\langle \alpha, \beta \rangle \in {}^{\omega}2 \times {}^{\omega}2 \mid \alpha \neq \beta\} \to {}^{\omega}\omega$$

such that

$$\alpha, \beta \in {}^{\omega}2, \quad \alpha \neq \beta \to (\forall n \in \omega)\ R(\overline{H(\alpha)}(n), \overline{H(\beta)}(n), \overline{K(\alpha, \beta)}(n)).$$

(Recall that $\overline{H(\alpha)}(n) = \langle H(\alpha)(0), H(\alpha)(1), \ldots, H(\alpha)(n-1) \rangle$.) Thus $\{H(\alpha) \mid \alpha \in {}^{\omega}2\}$ would be a perfect set of mutually inequivalent elements, contrary to assumption. Our method can be paraphrased by saying that we associate "witnesses" (i.e. γ's) to the failure of the E relation in a continuous manner.

Is it plausible to expect that we can associate the witnesses in a continuous fashion? This is answered affirmatively by the following statement.

Proposition. *Suppose Q is a binary relation on a subset of ${}^{\omega}\omega$ given by*

$$\alpha Q \beta \leftrightarrow (\exists \gamma \in {}^{\omega}\omega)\ W(\alpha, \beta, \gamma)$$

where W is analytic. If there is a perfect set P such that

$$\alpha, \beta \in P, \quad \alpha \neq \beta \to \alpha Q \beta,$$

then there are continuous functions

$$H : {}^{\omega}2 \to {}^{\omega}\omega, \qquad K : \{\langle \alpha, \beta \rangle \in {}^{\omega}2 \times {}^{\omega}2 \mid \alpha \neq \beta\} \to {}^{\omega}\omega$$

such that

$$\alpha, \beta \in {}^{\omega}2, \qquad \alpha \neq \beta \to W(H(\alpha), H(\beta), K(\alpha, \beta)).$$

The proof of this proposition is to be carried out by means of Lemma 2.3 and Corollary 2.4. The proof of Lemma 2.3 is presented in some detail, since Lemma 2.3 plays an essential part in Section 4. Easier proofs of Lemma 2.3 (at least for the case $\chi = {}^{\omega}\omega$) and Corollary 2.4 can be given using metamathematical techniques.

Let χ be a topological space. A set $S \subseteq \chi$ is said to be *comeager* iff there are sets $U_i \subseteq \chi$, each open dense in χ, such that $S \supseteq \bigcap_{i \in \omega} U_i$. A set $Y \subseteq \chi$ has the property of *Baire* iff there is an open $U \subseteq \chi$ and a comeager $S \subseteq \chi$ such that $Y \cap S = U \cap S$. It is well-known (see Oxtoby [9]) that the collection of sets having the property of Baire is closed under complementation and countable unions and intersections. A set is *meager* iff its complement is comeager. A set is *comeager* *relative to* U iff its intersection with U is comeager in the subspace U of χ (with the relative topology), or, what is equivalent, the complement of the set relative to U is meager. It is almost immediate from the definition of "property of Baire" that, if Y has the property of Baire and $Y \cap U$ is not meager, U open, then there is an open set $V \subseteq U$ such that Y is comeager relative to V.

Lemma 2.3. *Let χ be a Baire space (i.e., a topological space having no meager non-empty open sets), and suppose C is a closed subset of $\chi \times \omega^\omega$ such that $\pi_1(C) = \{x \in \chi \mid (\exists\alpha)\, \langle x, \alpha\rangle \in C\}$ is comeager. Then there is a comeager subset $Y \subseteq \chi$ and a Cohen continuous function $J : \chi \to \mathcal{P}$ such that, for all $x \in Y$, $\langle x, J(x)\rangle \in C$. (Instead of referring to a Cohen continuity, we could say that $J : Y \to \omega^\omega$ is continuous.) We may assume that $Y = \{x \in \chi \mid J(x) \in \omega^\omega\}$.*

Proof. We first define an auxiliary collection T and an auxiliary notion $X(U, s)$ where U open $\subseteq \chi$, $s \in {}^{<\omega}\omega$.

$$X(U, s) = \{x \in U \mid \exists\langle x, \alpha\rangle \in C \text{ where } \alpha \supseteq s\}$$

and

$$\langle U, s\rangle \in T \leftrightarrow U \text{ is open, nonempty in } \chi, s \in {}^{<\omega}\omega, \text{ and } X(U, s) \text{ is comeager relative to } U.$$

$\langle U, s\rangle$ and $\langle U', s'\rangle$ are said to be disjoint if $U \cap U' = \emptyset$. Put $\langle U, s\rangle \in T_n \leftrightarrow \langle U, s\rangle \in T$ and $s \in {}^n\omega$. We will define, for each n, a pairwise disjoint subcollection C_n of T_n such that

(i) $\bigcup\{U \mid \exists s \langle U, s\rangle \in C_n\}$ is dense in χ

and

(ii) If $n < m$, $\langle U, s\rangle \in C_n$, $\langle U', s'\rangle \in C_m$, then $U' \subseteq U$ and $s' \supseteq s$, or $U' \cap U = 0$.

Given such C_n's, define $J : \chi \to \mathcal{P}$ by the condition:

$$J(x)(n) = k \leftrightarrow \exists\langle U, s\rangle \in C_{n+1} \text{ such that } x \in U \text{ and } s(n) = k.$$

Clearly J is Cohen continuous. Let $Y = \{x \mid J(x) \in {}^\omega\omega\}$. Y is comeager. If $x \in Y$, $n \in \omega$, U an open set containing x, then we can find $\langle y, \alpha\rangle \in C$ such that $y \in U$, $\bar{\alpha}(n) = \overline{J(x)}(n)$ (so, by the closedness of C, $\langle x, J(x)\rangle \in C$): let $\langle U', s\rangle \in C_n$ be such that $x \in U'$, and take $y \in U \cap X(U', s)$ (possible because $X(U', s)$ is comeager in U', $U \cap U'$ open and nonempty in U') and α such that $\langle y, \alpha\rangle \in C$—recall the definition of $X(U', s)$).

The definition of the C_n's goes by induction. First observe that if $\langle U, s\rangle \in T$, $s \in {}^n\omega$, V open $\subseteq U$, then $\exists Z \subseteq V$, $t \supseteq s$, $t \in {}^{n+1}\omega$ such that $\langle Z, t\rangle \in T$. To prove this, note that $X(V, s)$ is not meager because $X(V, s) = X(U, s) \cap V$ and, since $\langle U, s\rangle \in T$, $X(U, s)$ is comeager relative to U. But

$$X(V, s) = \bigcup_{\substack{t \in {}^{n+1}\omega \\ t \supseteq s}} X(V, t).$$

So some such $X(V, t)$ is not meager (the union of countably many meager sets being meager). Hence, as was remarked above, there is an open $Z \subseteq V$ such that $X(Z, t) = Z \cap X(V, t)$ is comeager relative to Z.

Now given C_n, form, for each $\langle U, s\rangle \in C_n$, a maximal disjointed collection of $\langle Z, t\rangle \in T_{n+1}$ such that $Z \subseteq U$, $t \supseteq s$. Let T_{n+1} be the union of these collections as $\langle U, s\rangle$ ranges over C_n. Since $\bigcup \{U \mid \exists s\, \langle U, s\rangle \in C_n\}$ is dense, the remark made in the preceding paragraph yields immediately that $\bigcup \{Z \mid \exists s\, \langle Z, s\rangle \in C_{n+1}\}$ is dense.

No effort is made to state Corollary 2.4 in full generality.

Corollary 2.4. *Let C be a closed subset of ${}^{\omega}2 \times {}^{\omega}2 \times {}^{\omega}\omega$, such that*

$$(\forall \alpha, \beta \in {}^{\omega}2)(\exists \gamma \in {}^{\omega}\omega)(\langle \alpha, \beta, \gamma\rangle \in C).$$

Then there is a perfect set P and a continuous function $J : P \times P - \{\langle \alpha, \alpha\rangle \mid \alpha \in P\} \to {}^{\omega}\omega$ such that

$$(\forall \alpha, \beta \in P)(\alpha \neq \beta \to \langle \alpha, \beta, J(\alpha, \beta)\rangle \in C).$$

Outline of proof. By Lemma 2.3, there is a Cohen continuous $J' : {}^{\omega}2 \times {}^{\omega}2 \to \mathcal{P}$ and a comeager set Y such that $\langle \alpha, \beta\rangle \in Y \to \langle \alpha, \beta, J(\alpha, \beta)\rangle \in C$. It remains only to show that any comeager subset Y of ${}^{\omega}2 \times {}^{\omega}2$ includes a set of the form

$$P \times P - \{\langle \alpha, \alpha\rangle \mid \alpha \in P\},$$

P perfect, for then one may take $J = J'$ restricted to $P \times P - \{\langle \alpha, \alpha\rangle \mid \alpha \in P\}$.

We may assume $Y = \bigcap_{n \in \omega} U_n$ where each U_n is open dense. Form a perfect tree (see Section 1 for the definitions) $T \subseteq {}^{\omega}2$ and numbers $l_0 < l_1 < l_2 < \cdots$ such that, whenever $p, q \in {}^{l_n}2$ are in T, $p \neq q$, then $[p, q] \subseteq U_n$. One defines l_n and $T \cap {}^{l_n}2$ by induction on n: at each stage, a number of refinements are necessary since several different conditions are being imposed on $T \cap {}^{l_n}2$. It is easy to check that, if $P =$ the set of paths through T, then $P \times P - \{\langle \alpha, \alpha\rangle \mid \alpha \in P\} \subseteq Y$. (This argument is well-known.)

Finally, it is easy to reduce the proposition stated above to Corollary 2.4. First let $M : {}^{\omega}2 \to P$ be 1–1 continuous. Then write

$$\{\langle \alpha, \beta, \gamma\rangle \mid \alpha = \beta \vee W(M(\alpha), M(\beta), \gamma\}$$

as the projection of a closed set and apply Corollary 2.4.

3. The first part of the proof

In this section, we will be making use of the axiom of choice and the generalized continuum hypothesis (GCH). The GCH is used only to simplify some statements and calculations which might otherwise prove disagreeable. The non-logician who wishes to eliminate the GCH in the most straightforward way possible can simply go through the arguments of this section and replace each reference to $\aleph_\eta$ by a reference to $\beth_\eta$ or some slightly more complicated formulation involving the $\beth$ symbol (where one defines: $\beth_0 = \aleph_0$, $\beth_{\eta+1} = 2^{\beth_\eta}$, and, if λ is a limit ordinal, $\beth_\lambda = \bigcap_{\eta>\lambda} \beth_\eta$). The key point is that there is a variant of Proposition 3.10 which doesn't depend on the GCH.

However, there is a logician's trick which enables one to dispense with the axiom of choice and GCH at one stroke without further ado. It is a folklore result that any Π^1_3 sentence of number theory provable in ZF + AC + GCH is already provable in ZF. (Sketch of proof: If the sentence in question is $(\forall\alpha)(\exists\beta)(\forall\gamma)\ \varphi(\alpha, \beta, \gamma)$, where φ is first-order, then, for each $\alpha \in {}^\omega\omega$, since $L[\alpha] \models$ ZF + AC + GCH, our hypothesis implies that $L[\alpha] \models (\exists\beta)(\forall\gamma)\ \varphi(\alpha, \beta, \gamma)$, so by Shoenfield's absoluteness theorem [10], $(\exists\beta)(\forall\gamma)\ \varphi(\alpha, \beta, \gamma)$. Since $\alpha \in {}^\omega\omega$ was arbitrary, we have $(\forall\alpha)(\exists\beta)(\forall\gamma)\varphi(\alpha, \beta, \gamma)$.) In virtue of the discussion in the preceding section, our main theorem can be reformulated as Π^1_3 sentence of number theory:

> ($\forall$ closed $C \subseteq {}^\omega\omega \times {}^\omega\omega \times {}^\omega\omega$) [If the relation E given by $E(\alpha, \beta) \leftrightarrow \forall\gamma \neg C(\alpha, \beta, \gamma)$ is an equivalence relation on ${}^\omega\omega$ having uncountably many equivalence classes, then there are continuous functions $H: {}^\omega 2 \to {}^\omega\omega$, $K: {}^\omega 2 \times {}^\omega 2 - \{\langle\alpha, \alpha\rangle \mid \alpha \in {}^\omega 2\} \to {}^\omega\omega$ such that, whenever $\alpha \neq \beta$, $C(H(\alpha), H(\beta), K(\alpha, \beta))$].

Since closed sets can be coded by members of ${}^\omega\omega$, we may treat C in this expression as a variable ranging over ${}^\omega\omega$. The same applies to H and K. Now it is easily seen that the antecedent of the expression in brackets is Π^1_2, the consequent Σ^1_2, hence the entire expression is Π^1_3. The same sort of considerations may be used in connection with the theory: $Z + \exists$ uncountably many cardinals.

In this section and the next, we are assuming, by way of contradiction, that E is a coanalytic equivalence relation such that

(i) E has uncountably many equivalence classes,

(ii) there is no perfect set of mutually E inequivalent elements $\in {}^\omega\omega$, and

(iii) the field of E is precisely ${}^\omega\omega$, i.e. E is a relation on ${}^\omega\omega$ and $(\forall\alpha \in {}^\omega\omega)(\alpha E \alpha)$.

It is necessary, or at least expedient, to present a proof by contradiction, because assumption (ii) must be invoked twice, once in this section and once in Section 4.

Since E is coanalytic, we may, in view of the discussion in Section 2, assume

$$(\forall\alpha, \beta \in {}^\omega\omega)(\neg(\alpha E \beta) \leftrightarrow (\exists\gamma \in {}^\omega\omega)(\forall n \in \omega) R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)))$$

where R is a ternary relation on ${}^{<\omega}\omega$. For convenience, we write

$$S(\alpha, \beta, \gamma) \leftrightarrow (\forall n \in \omega)\ R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n))$$

so we have

$$\neg(\alpha E\beta) \leftrightarrow (\exists\gamma \in {}^\omega\omega)\, S(\alpha, \beta, \gamma).$$

It will be shown in the next section, on the basis of our assumptions concerning E, that one can obtain:

(#) sequences F_η, $\eta < \aleph_{\omega_1}$, and $G_{\eta\xi}$, $\eta < \xi < \aleph_{\omega_1}$,

and λ_η, $\eta < \aleph_{\omega_1}$, such that each F_η is a Cohen continuous function from ${}^\omega\lambda_\eta$ into $\mathcal{P}$, each $G_{\eta\xi}$ is a Cohen continuous function from ${}^\omega\lambda_\eta \times {}^\omega\lambda_\xi$ into $\mathcal{P}$, each λ_η is an ordinal, and: whenever $\eta < \xi < \aleph_{\omega_1}$, $\alpha \in {}^\omega\lambda_\eta$, $\beta \in {}^\omega\lambda_\xi$, and $F_\eta(\alpha)$, $F_\xi(\beta)$, and $G_{\eta\xi}(\alpha, \beta)$ are members of ${}^\omega\omega$, then

$$S(F_\eta(a), F_\xi(\beta), G_{\eta\xi}(\alpha, \beta)).$$

Here ω_1 is the first uncountable ordinal, and $\aleph_{\omega_1}$ is the ω_1st infinite cardinal. Of course ${}^\omega\lambda_\eta$ is simply the set of functions from ω = the set of natural numbers into the ordinal λ_η (where, as with any ordinal, one identifies λ_η and the set of ordinals $< \lambda_\eta$).

For the benefit of logicians and set theorists, one may paraphrase (#) by saying that each F_η is a forcing term with respect to the partial ordering ${}^{\underset{\smile}{\omega}}\lambda_\eta$ appropriate for mapping ω onto λ_η such that F_η always denotes a member of ${}^\omega\omega$, and each $G_{\eta\xi}$ is a forcing term appropriate for ${}^{\underset{\smile}{\omega}}\lambda_\eta \times {}^{\underset{\smile}{\omega}}\lambda_\xi$ which always denotes a witness to the E inequivalence of $F_\eta(\alpha)$ and $F_\xi(\beta)$.

Lemma 3.1. *Suppose $p \in {}^{\underset{\smile}{\omega}}\lambda_\eta$, $q \in {}^{\underset{\smile}{\omega}}\lambda_\xi$ and $s, t, u \in {}^n\omega$ are such that $p \Vdash \bar{F}_\eta(n) = s$, $q \Vdash \bar{F}_\xi(n) = t$, and $\langle p, q\rangle \Vdash \bar{G}_{\eta\xi}(n) = u$. Then $R(s, t, u)$.*

Proof. Let p, q, s, t, u be as above. By the Cohen continuity of F_η, F_ξ, $G_{\eta\xi}$, it is easy to find $\alpha \in {}^\omega\lambda_\eta$, $\beta \in {}^\omega\lambda_\xi$, $\alpha \supseteq p$, $\beta \supseteq q$ such that $F_\eta(\alpha)$, $F_\xi(\beta)$, $G_\eta\xi(\alpha, \beta)$ are all members of ${}^\omega\omega$. (We just make sure that α, β, and $\langle\alpha, \beta\rangle$ lie in all the relevant open dense sets, which can be done since there are only countably many.) By (#), $S(F_\eta(\alpha), F_\xi(\beta), G_{\eta\xi}(\alpha, \beta))$. By the definition of S, this means in particular: $R(\overline{F_\eta(\alpha)}(n), \overline{F_\xi(\beta)}(n), \overline{G_{\eta\xi}(\alpha, \beta)}(n))$. But. $p \Vdash \overline{F_\eta(n)} = s$ and $\alpha \supseteq p$ imply that $\overline{F_\eta(\alpha)}(n) = s$. Similarly, $\overline{F_\xi(\beta)}(n) = t$, $\overline{G_{\eta\xi}(n)} = u$.

I have chosen to prove Lemma 3.1 from (#) instead of incorporating it into the statement of (#) in order to keep the statement of (#) relatively simple.

We propose to obtain a contradiction by showing that, contrary to hypothesis, there is a perfect set of mutually E inequivalent members of ${}^\omega\omega$. We shall actually obtain continuous functions

$$H : {}^\omega 2 \to {}^\omega\omega, \quad K : \{\langle\alpha, \beta\rangle \in {}^\omega 2 \times {}^\omega 2 \mid \alpha < \beta\} \to {}^\omega\omega$$

such that

$$(\forall\alpha, \beta \in {}^\omega 2)(\alpha < \beta \to S(H(\alpha), H(\beta), K(\alpha, \beta)).$$

(Here $<$ is the lexiographical ordering of ${}^{\omega}2$, i.e., $\alpha<\beta$ iff $(\exists n\in\omega)$ $\alpha(0)=\beta(0)\wedge\alpha(1)=\beta(1)\wedge\cdots\wedge\alpha(n-1)=\beta(n-1)\wedge\alpha(n)<\beta(n)$). In fact, H and K will be strongly continuous in the sense that the value of $H(\alpha)(n-1)$ depends only on $\bar{\alpha}(n)$, and of $K(\alpha,\beta)(n-1)$ only on $\langle\bar{\alpha}(m),\bar{\beta}(m)\rangle$ where m is the least number $\geqslant n$ such that $\bar{\alpha}(m)\neq\bar{\beta}(m)$. We may express this formally by saying (after saying that H and K are continuous):

$$\text{if } p\Vdash H(n-1)=m,\ \text{then } \bar{p}(n)\Vdash H(n-1)=m$$

and

$$\text{if } \langle p,q\rangle\Vdash K(n-1)=j,\ \text{and } m\geqslant n,\qquad \bar{p}(m)\neq\bar{q}(m),$$

then

$$\langle\bar{p}(m),\bar{q}(m)\rangle\Vdash K(n-1)=j.$$

Now such a pair H,K of "strongly continuous" functions can be viewed as being built up by stages. H,K are completely specified by the relations $\Vdash_H$, $\Vdash_K$. For each n, let M_n be the pair $\langle M_n^1, M_n^2\rangle$ where M_n^1 is the result of restricting $\Vdash_H$ to $\bigcup_{i\leqslant n}{}^i2$ and M_n^2 is the result of restricting $\Vdash_K$ to $(\bigcup_{i\leqslant n}{}^i2)\times(\bigcup_{i\leqslant n}{}^i2)$. The pair $\langle\Vdash_H,\Vdash_K\rangle$ may be viewed as $\bigcup_{n\in\omega}M_n$. Accordingly, we will construct H,K by defining a sequence $M_0, M_1, M_2,\ldots$. In the definition about to be stated, $<$ is always the relevant lexicograpical ordering: if $x,y\in{}^i2$, then $x<y$ iff there is $j<i$ such that $x(j)<y(j)$ while $(\forall l<j)\ x(l)=y(l)$.

Definition 3.2. *M is an n-sprout* iff $M=\langle M^1, M^2\rangle$ where M^1 is a binary relation between members of $\bigcup_{i\leqslant n}{}^i2$ and sentences $H(i)=m$ (for $i<n$) and M^2 is a binary relation between members of $\bigcup_{i\leqslant n}{}^i2\times\bigcup_{i\leqslant n}{}^i2$ and sentences $K(i)=m$, $i<n$, for which the following conditions are satisfied:

(a) If $0<i<n$, $x,y\in{}^i2$, then there are a and $b\in\omega$ such that

$$xM^1[H(i-l)=a]$$

and, if $x<y$, $\langle x,y\rangle M^2[K(i-1)=b]$. Moreover a, and, if $x<y$, b, are unique.

(b) If σ is a sentence of one of the relevant forms ($H(\gamma)(i)=a$ or $K(i)=b$, $i<n$), and $xM^1\ \sigma$ *and* x^1 extends x, $x^1\in\bigcup_{i\leqslant n}{}^i2$, then $x^1\ M^1\ \sigma$. Similarly for $\langle x,y\rangle$ and M^2.

(c) (This is both a separate definition and by way of preparation for stating condition (d).) Let $M^1[x]$ be that sequence $s\in{}^n\omega$ such that, for all $i<n$,

$$xM^1[H(i)=s(i)],$$

and, for $x,y\in{}^n2$, $x<y$, $M^2[x,y]=$ that $u\in{}^n\omega$ such that, for all $i<n$, $\langle x,y\rangle M^2[K(i)=u(i)]$.

(d) Given the definitions of $M^1[x]$ and $M^2[x,y]$ in (c), we require:

$$\text{If } x<y \text{ are in } {}^n2 \text{ and } i\leqslant n,\ \text{then}$$
$$R(\overline{M^1[x]}(i),\ \overline{M^1[y]}(i),\ \overline{M^2[x,y]}(i)).$$

Thus an n-sprout is an "nth approximation" to a pair of strongly continuous functions H,K with the properties described above.

Definition 3.3. Suppose $n < n'$, M and N are n and n' sprouts respectively. We say that N extends M if and only if the restriction of N^1 to $\bigcup_{i<n} {}^i2$ is M^1, similarly for N^2 and M^2. (It is not hard to show that this is equivalent to: if $x \in {}^n2$, $x' \in {}^{n'}2$, $x \subseteq x'$, then $M^1[x] \subseteq N^1[x']$; and, if $x, y \in {}^n2$, $x < y$, $x \subseteq x'$, $y \subseteq y'$, $x', y' \in {}^{n'}2$, then $M^1[x, y] \subseteq N^1[x', y']$.)

The following lemma is almost immediate.

Lemma 3.4. *Suppose $M_0, M_1, M_2, \ldots$ is a sequence such that each M_n is an n-sprout and, if $n < n'$, then $M_{n'}$ extends M_n. Then there exist continuous functions $H: {}^\omega 2 \to {}^\omega\omega$ and $K: \{\langle\alpha, \beta\rangle \in {}^\omega 2 \times {}^\omega 2 \mid \alpha < \beta\} \to {}^\omega\omega$ such that $(\forall \alpha, \beta \in {}^\omega 2)(\alpha < \beta \to S(H(\alpha), H(\beta), K(\alpha, \beta)))$, specified by the requirements*

$$\Vdash_H \supseteq \bigcup_{n \in \omega} M^1_n, \qquad \Vdash_K \supseteq \bigcup_{n \in \omega} M^2_n.$$

(*One defines:* $H(\alpha)(m) = n \leftrightarrow (\exists p \subseteq \alpha)\, p \Vdash H(m) = n$ *and* $K(\alpha, \beta) = n \leftrightarrow (\exists p \subseteq \alpha, q \subseteq \beta)\langle p, q\rangle \Vdash K(m) = n$.)

Accordingly, we are reduced to defining a sequence M_0, M_1, satisfying the hypothesis of lemma 3.4. The idea is to find a notion of "viable n-sprout" such that, whenever M is a viable n-sprout, there is a viable $n+1$-sprout extending it, and such that the trivial sprout is viable. Then it is easy to obtain, inductively, a sequence $M_0, M_1, \ldots$ of viable n-sprouts.

Our concept of viability is reminiscent of (and, indeed, was influenced by)[3] a technique employed in model theory by Morley [7]. We need a preliminary definition first.

Definition 3.5. Fix $\eta < \aleph_{\omega_1}$. We define a sequence of equivalence relations $\sim_\tau$, $\tau < \omega_1$, on ${}^\omega\lambda_\eta$ by induction on τ. In each case, $[p]_\tau = \{q \mid q \sim_\tau p\}$.

(i) $p \sim_0 q \leftrightarrow (\forall m, n)(p \Vdash F_\eta(m) = n \leftrightarrow q \Vdash F_\eta(m) = n)$.
(ii) $p \sim_{\tau+1} q \leftrightarrow \{[s]_\tau \mid s \in {}^\omega\lambda_\eta,\ s \text{ extends } p\} = \{[s]_\tau \mid s \in {}^\omega\lambda_\eta,\ s \text{ extends } q\}$.
(iii) If ν is a limit ordinal, $p \sim_\nu q \leftrightarrow (\forall \tau < \nu)(p \sim_\tau q)$.

An easy induction (using GCH) shows that the number of $\sim_\tau$ equivalence classes is at most $\aleph_{\tau+1}$: For example, suppose the claim is true for τ. If $p \in {}^\omega\lambda_\eta$, let $h(p) = \{[q]_\tau \mid q$ extends $p\}$. From the definition of $\sim_{\tau+1}$,

$$p \underset{\tau+1}{\sim} p' \quad \text{iff } h(p) = h(p').$$

But every $h(p)$ is a subset of

$$\{[q]_\tau \mid q \in {}^\omega\lambda_\eta\},$$

[3] In other respects, our proof has been influenced by Solovay [11].

which, by induction hypothesis, has cardinality $\leqslant \aleph_{\tau+1}$. Hence $\{h(p) \mid p \in {}^{\omega}\lambda_\eta\}$ has cardinality at most $2^{\aleph_{\tau+1}} = \aleph_{\tau+2}$ (invoking the GCH). In other words, there are at most $\aleph_{\tau+2}$ $\sim_{\tau+1}$ equivalence classes. To handle the case where ν is a limit ordinal, define, for each $p \in {}^{\omega}\lambda_\eta$,

$$g(p) = \langle [p]_\tau \mid \tau < \nu \rangle$$

(i.e. $g(p)$ is the function with domain ν which assigns to each $\tau < \nu$ the value $[p]_\tau$). Since $p \sim_\nu p'$ if and only if $g(p) = g(p')$, there is a 1–1 mapping of the set of $\sim_\nu$ equivalence classes into $\mathsf{X}_{\tau<\nu} Y_\tau$ where $Y_\tau =$ the set of $\sim_\tau$ equivalence classes. By induction hypothesis, each Y_τ has cardinality $\leqslant \aleph_{\tau+1} < \aleph_\nu$. So the number of $\sim_\nu$ equivalence classes is at most $\aleph_\nu^\nu$ which, by GCH, is at most $\aleph_{\nu+1}$.

Definition 3.6. (a) A (μ, ν) *system* of length ${}^n 2$ (where, as usual, ${}^n 2$ is the set of binary sequences of length n) is a sequence

$$\langle J, X_t \rangle_{t \in {}^n 2}$$

for which the following three conditions hold:

(i) For each $t \in {}^n 2$, X_t is a subset of $\aleph_{\omega_1}$ having cardinality at least $\aleph_\mu$.

(ii) If $t_1 < t_2$ in the lexicographical ordering of ${}^n 2$ and $\eta_1 \in X_{t_1}$, $\eta_2 \in X_{t_2}$, then $\eta_1 < \eta_2$.

(iii) Let $X = \bigcup_{t \in {}^n 2} X_t$. Then J is a function whose domain is X which assigns to each $\eta \in X$ an $\sim_\nu$ equivalence class in ${}^{\omega}(\lambda_\eta)$.

(b) Suppose $M = \langle M^1, M^2 \rangle$ is an n-sprout. The (μ, ν) system of length ${}^n 2$ mentioned in (a) is said to *satisfy* M if and only if, whenever $t_1 < t_2$ are members of ${}^n 2$ ($<$ being the lexicographical ordering of ${}^n 2$) and $\eta_1 \in X_{t_1}$, $\eta_2 \in X_{t_2}$, then there are $p_{\eta_1\eta_2} \in {}^{\omega}(\lambda_{\eta_1})$, $q_{\eta_1\eta_2} \in {}^{\omega}(\lambda_{\eta_2})$ such that the following conditions hold:

(i) The ν-equivalence class of $p_{\eta_1\eta_2}$ is $J(\eta_1)$ and the ν-equivalence class of $q_{\eta_1\eta_2}$ is $J(\eta_2)$.

(ii) $\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2} \rangle \Vdash \bar{G}_{\eta_1\eta_2}(n) = M^2[t_1, t_2]$.

(iii) $p_{\eta_1\eta_2} \Vdash \bar{F}_{\eta_1}(n) = M^1[t_1]$ and $q_{\eta_1\eta_2} \Vdash \bar{F}_{\eta_2}(n) = M^1[t_2]$.

(c) Let $X = \bigcup_{t \in {}^n 2} X_t$. A sequence $\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2} \mid \eta_1 < \eta_2$ in $X \rangle$ is said to be a *witness to* $\langle J, X_t \rangle_{t \in {}^n 2}$ *satisfying* M if and only if every $p_{\eta_1\eta_2} \in {}^{\omega}\lambda_{\eta_1}$, $q_{\eta_1\eta_2} \in {}^{\omega}\lambda_{\eta_2}$, and the p's and q's satisfy conditions (i)–(iii) in (b) above, (i) being required whenever $\eta_1 < \eta_2$ and (ii) and (iii) only when $\eta_1 \in X_{t_1}$, $\eta_2 \in X_{t_2}$, $t_1 < t_2$.

Let me clarify (ii) and (iii) in Definition 3.6(b) a bit. For example, (ii) can be reformulated in the following way. Let s be the sequence $M^2[t_1, t_2]$. Recall that $s \in {}^n\omega$. (ii) means that, for every $i < n$,

$$\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2} \rangle \Vdash G_{\eta_1\eta_2}(i) = s(i).$$

A similar explication can be given for (iii).

Remark. It is immediate that, if $\mu < \mu'$, then any (μ', ν) system is also a (μ, ν) system. Also, if $\langle J, X_t \rangle_{t \in {}^n 2}$ is a (μ, ν') system satisfying M and $\nu < \nu'$, then in a natural way one obtains a (μ, ν) system $\langle J, X_t\rangle_{t\in {}^n 2}$ satisfying M. Just take

> $J(\eta)$ = the ν-equivalence class which includes the ν'-equivalence class $J(\eta)$.

The same witnesses work.

Definition 3.7. An n-sprout $M=\langle M^1, M^2\rangle$ is *viable* if and only if, for every $\mu<\omega_1$, there is a (μ, μ) system satisfying M.

In view of the remark just preceding this definition, it is clear that an n-sprout M is viable if and only if it is satisfied by (μ, μ) systems for μ cofinal in ω_1.

To complete this part of the proof, it now sufficies to show that any viable n-sprout can be extended to a viable $(n+1)$-sprout.

Main Lemma 3.8. *If $\langle J', X'_t\rangle_{t\in {}^n 2}$ is a $(\mu+2^{2n+3}, \mu+2^{2n+3})$ system satisfying the n-sprout M', then there is a (μ, μ) system $\langle J, X_t\rangle_{t\in {}^{n+1} 2}$ satisfying some $(n+1)$-sprout M which extends M'. (In fact, this can be done so that $X_{t^\frown\langle 0\rangle}$ and $X_{t^\frown\langle 1\rangle}$ are subsets of X_t.)*

Corollary 3.9. *Any viable n-sprout can be extended to a viable $(n+1)$-sprout.*

Proof of Corollary from Main Lemma. Let M' be a viable n-sprout. For each $\mu<\omega_1$, there is, owing to the viability of M', a $(\mu+2^{2n+3}, \mu+2^{2n+3})$ system satisfying M' and hence, by the Main Lemma, a (μ, μ) system satisfying some $(n+1)$-sprout M_μ which extends M'.

Since there are only countably many n-sprouts, there is an uncountable set $C\subseteq\omega_1$ such that M_η is the same for all $\eta\in C$. Let M be the common value of the M_η. From the "Remark" in the second paragraph after Definition 3.6, it is clear that, for each $\mu<\omega_1$, there is a (μ, μ) system satisfying M, i.e. M is viable. Hence M is a viable $(n+1)$-sprout extending M'.

Before turning to the proof of the Main Lemma, it is useful to state and prove the following well-known theorem from the theory of polarized partition relations. For simplicity, we state it in a form which presupposes the GCH. Recall that the GCH is the assertion that, for all η, $2^{\aleph_\eta}=\aleph_{\eta+1}$.

Proposition 3.10. (GCH) [12]. *Let κ be a cardinal and $f:\kappa^{+++}\times\kappa^+\to\kappa$. Then there are subsets X, Y of κ^{+++}, κ^+, respectively, such that f is constant on $X\times Y$ and* card $X=\kappa^{+++}$, *and $Y=\kappa^+$. (Here κ^+ = the least cardinal $>\kappa$. Thus, if $\kappa=\aleph_\eta$, then $\kappa^+=\aleph_{\eta+1}$ and $\kappa^{+++}=\aleph_{\eta+3}$.)*

Proof. For each $\eta\in\kappa^{+++}$, let $f_\eta:\kappa^+\to\kappa$ be given by

$$f_\eta(\xi)=f(\eta, \xi).$$

By GCH, there are at most κ^{++} functions from κ^{+} into κ. So there is a subset X of κ^{+++} having cardinality κ^{+++} such that f_η is the same for all $\eta \in X$. Let g be the common value. Since $g : \kappa^{+} \to \kappa$, there is $Y \in \kappa^{+}$ of cardinality κ^{+} on which g is constant. Clearly f is constant on $X \times Y$.

We turn now to the proof of the Main Lemma 3.8. We are given a $(\mu + 2^{2n+3}, \mu + 2^{2n+3})$ system $\langle J', X'_t\rangle_{t\in {}^n2}$ which satisfies a certain n-sprout M'. Let $X' = \bigcup_{t\in {}^n2} X'_t$ and let $\langle p'_{\eta\eta'}, q'_{\eta\eta'} \mid \eta < \eta' \text{ in } X'\rangle$ be a witness to $\langle J', X'_t\rangle_{t\in {}^n2}$ satisfying M' (see Definition 3.6). It is our aim to find a (μ, μ) system satisfying some $(n+1)$-sprout M extending M'. To achieve this aim, it will suffice to find, for some positive integer u,

3.11. a $(\mu + u, \mu + u)$ system $\langle J, X_t\rangle_{t\in {}^{(n+1)}2}$, an $(n+1)$-sprout M, and a sequence $\langle p_{\eta\eta'}, q_{\eta\eta'} \mid \eta < \eta' \text{ in } X\rangle$ (where $X = \bigcup_{t\in {}^{(n+1)}2} X_t$) which is a witness to $\langle J, X_t\rangle_{t\in {}^{(n+1)}2}$ satisfying M such that

(a) for all $t \in {}^n2$, both $X_{t\frown\langle 0\rangle}$ and $X_{t\frown\langle 1\rangle}$ are subsets of X'_t, and

(b) whenever $t_1 < t_2$ are in ${}^n2$ and $\eta_1 \in X \cap X'_{t_1}$, $\eta_2 \in X \cap X'_{t_2}$ extends $p'_{\eta_1\eta_2}$ and then $q_{\eta_1\eta_2}$ extends $q'_{\eta_1\eta_2}$.

Why is 3.11 sufficient to prove Main Lemma 3.8? Conditions (a) and (b) insure that M extends M'. For example, if $t_1 < t_2$ are in ${}^n2$ and i, j are 0 or 1, we need to see that $M^2[t_1\frown\langle i\rangle, t_2\frown\langle j\rangle]$ extends $M'^2[t_1, t_2]$. (See remark after Definition 3.3.) Let

$$\eta_1 \in X_{t_1\frown\langle i\rangle}, \quad \eta_2 \in X_{t_2\frown\langle j\rangle}.$$

Thus $\eta_1 \in X \cap X'_{t_1}$ and $\eta_2 \in X \cap X'_{t_2}$, by (a). Thus, by (b), $p_{\eta_1\eta_2}$ extends $p'_{\eta_1\eta_2}$, $q_{\eta_1\eta_2}$ extends $q'_{\eta_1\eta_2}$. By the defining property of the p', q' sequence,

$$\langle p'_{\eta_1\eta_2}, q'_{\eta_1\eta_2}\rangle \Vdash \overline{G_{\eta_1\eta_2}}(n) = M'^2[t_1, t_2].$$

By 3.11,

$$\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2}\rangle \Vdash \overline{G_{\eta_1\eta_2}}(n+1) = M^2[t_1\frown\langle i\rangle, t_2\frown\langle j\rangle].$$

Since $p_{\eta_1\eta_2}$ extends $p'_{\eta_1\eta_2}$ and $q_{\eta_1\eta_2}$ extends $q'_{\eta_1\eta_2}$, an elementary property of $\Vdash$ insures that $M^2[t_1\frown\langle i\rangle, t_2\frown\langle j\rangle]$ extends $M'^2[t_1, t_2]$, as desired. Thus there is an $(n+1)$-sprout M extending M' which (i.e., M) is satisfied by a $(\mu + u, \mu + u)$ system, u some positive integer. By the Remark after Definition 3.6, M is satisfied by some (μ, μ) system. Thus 3.11 is sufficient to prove Main Lemma 3.8.

Here is our strategy to prove 3.11. First, for each $\eta \in X'$ choose a $\mu + 2^{2n+3} - 1$ equivalence class "beneath" $J'(\eta)$ which decides $F_\eta(n)$. More precisely, given $\eta \in X'$, choose an η' in a different block from η (i.e. in a different X'_t). For the sake of definiteness, say $\eta < \eta'$ (in the other case, just consider $q'_{\eta'\eta}$ instead of $p'_{\eta\eta'}$). By an elementary property of Cohen continuity, there is $p''_{\eta\eta'}$ extending $p'_{\eta\eta'}$

in ${}^{\omega}(\lambda_\eta)$ and a natural number a such that

$$p''_{\eta'} \Vdash F_\eta(n) = a.$$

Since $p'_{\eta\eta'}$ decides $F_\eta(n)$, $p''_{\eta\eta'}$ decides $\bar{F}_\eta(n+1)$, say

$$p''_{\eta\eta'} \Vdash \bar{F}_\eta(n+1) = s_\eta,$$

where $s_\eta \in {}^{\omega}\omega$. For any ξ such that $p'_{\eta\xi}$ is defined, one may find $p''_{\eta\xi}$ $\mu + 2^{2n+3} - 1$ equivalent to $p''_{\eta\eta'}$ which extends $p'_{\eta\xi}$ (this is possible — see Definition 3.5 — because $p'_{\eta\xi}$ is $\mu + 2^{2n+3}$ equivalent to $p'_{\eta\eta'}$, the latter being true because the p', q' sequences witness to $\langle J', X'_t\rangle$ satisfying M — see Definition 3.6(c)). Choose $q''_{\xi\eta}$ in the same way if $\xi < \eta$. It is in this sense that we have chosen a $\mu + 2^{n+3} - 1$ equivalence class beneath $J'(\eta)$: viz., the $\mu + 2^{n+3} - 1$ equivalence class shared by all $p''_{\eta\xi}$ and $q''_{\xi\eta}$, η fixed. Now stabilize s_η: For each $t \in {}^{n}2$, let X''_t be a subset of X'_t of cardinality $\aleph_{\mu+2^{2n+3}}$ such that s_η is the same for all $\eta \in X''_t$. (This is possible since ${}^{\omega}\omega$, the set from which the s_η come, is countable, while X'_t is a set of regular cardinality $\aleph_{\mu+2^{2n+3}}$.) Finally, for each $t \in {}^{n}2$, let $X^0_{t\frown\langle 0\rangle}$, $X^0_{t\frown\langle 1\rangle}$ be subsets of X''_t, each of cardinality $\aleph_{\mu+2^{2n+3}-1}$, such that every element of $X^0_{t\frown\langle 0\rangle}$ is less than every element of $X^0_{t\frown\langle 1\rangle}$. (This can be done since X_t has cardinality $\aleph_{\mu+2^{2n+3}}$.) Let $X^0 = \bigcup_{t\in{}^{(n+1)}2} X^0_t$, and put $p^0_{\eta\xi} = p''_{\eta\xi}$, $q^0_{\eta\xi} = q''_{\eta\xi}$ whenever $\eta < \xi$ are in X^0.

To avoid cumbersome notation, let ν be the ordinal $\mu + 2^{2n+3} - 1$. Thus we have sets X^0_t, $t \in {}^{(n+1)}2$, each of cardinality $\aleph_\nu$, as well as the $p^0_{\eta\xi}$, $q^0_{\xi\eta}$ mentioned above, where, for fixed η, all of the $p^0_{\eta\xi}$ and $q^0_{\xi\eta}$ are ν-equivalent. Note that there are $2^{n+1}(2^{n+1}-1)/2 = 2^n(2^{n+1}-1)$ pairs of the form $\langle t, t'\rangle$ where $t, t' \in {}^{(n+1)}2$ and $t < t'$ (lexicographically). *Let* $k = 2^n(2^{n+1}-1)$, *and let* $\langle t_i, t'_i\rangle$, $i = 1, \ldots, k$, *enumerate all such pairs.*

It is now our intention to state 3.12, from which 3.11 directly follows, as will be seen.

3.12. There are sets X^i_t, $i = 1, \ldots, k$, $t \in {}^{(n+1)}2$, where each X^i_t is a subset of X^{i-1}_t (in particular, $X^1_t \subseteq X^0_t$, where X^0_t is as above) and sequences $\langle p^i_{\eta\xi}, q^i_{\eta\xi} \mid \eta < \xi$ in $X^i\rangle$ for $i = 1, \ldots, k$, where $X^i = \bigcup_{t\in{}^{(n+1)}2} X^i_t$, such that each X^i_t has cardinality $\aleph_{\nu-4i}$ and the following hold (for $i = 1, \ldots, k$):

(A) For fixed η, all $p^i_{\eta\xi}$ and $q^i_{\xi'\eta}$ are $\sim_{\nu-4i}$ equivalent (more precisely: all the $p^i_{\eta\xi}$ for fixed η are $\sim_{\nu-4i}$ equivalent among themselves, as are the $q^i_{\xi'\eta}$, and each $p^i_{\eta\xi}$ is $\sim_{\nu-4i}$ equivalent to $q^i_{\xi'\eta}$, η fixed).

(B) $p^i_{\eta\xi}$ extends $p^{i-1}_{\eta\xi}$ and $q^i_{\eta\xi}$ extends $q^{i-1}_{\eta\xi}$ whenever $\eta < \xi$ are in X^i.

(C) There is a sequence $s^i \in {}^{(n+1)}\omega$ such that, whenever $\eta \in X^i_{t_i}$, $\xi \in X^i_{t'_i}$, then

$$\langle p^i_{\eta\xi}, q^i_{\eta\xi}\rangle \Vdash \bar{G}_{\eta\xi}(n+1) = s^i.$$

Before proceeding to the proof of 3.12, we first indicate how to show that, by putting

$$X_t = X^k_t, \qquad p_{\eta\xi} = p^k_{\eta\xi}, \qquad q_{\eta\xi} = q^k_{\eta\xi},$$

we obtain from 3.12, objects satisfying 3.11, where

$$u = 2^{2n+3} - 1 - 4k = 2^{2+n} - 1$$

is a positive integer. One takes $J(\eta)$ to be the common $\mu + u = \nu - 4k$ equivalence class of all of the $p_{\eta\xi}$ and $q_{\xi\eta}$. Part (a) of 3.11 holds because, if $t \in {}^n2$ and $j = 0$ or 1, then

$$X_{t\frown\langle j\rangle} = X^k_{t\frown\langle j\rangle} \subseteq X^{k-1}_{t\frown\langle j\rangle} \subseteq \cdots \subseteq X^0_{t\frown\langle j\rangle} \subseteq X'_t.$$

(The notation $t\frown\langle j\rangle$ is defined by: if $t = \langle t_0, \ldots, t_{n-1}\rangle$, then $t\frown\langle j\rangle = \langle t_0, \ldots, t_{n-1}, j\rangle$.) Part (b) holds because

$$p_{\eta_1\eta_2} = p^k_{\eta_1\eta_2} \supseteq p^{k-1}_{\eta_1\eta_2} \supseteq \cdots \supseteq p^0_{\eta_1\eta_2} = p''_{\eta_1\eta_2} \supseteq p'_{\eta_1\eta_2}.$$

Finally, we must see that there is an $(n+1)$-sprout M such that $\langle p_{\eta\xi}, q_{\eta\xi} \mid \eta < \xi$ in $X\rangle$ witnesses to $\langle J, X_t\rangle_{t \in {}^{n+1}2}$ satisfying M.

For $0 \leqslant i \leqslant n$ and $y \in {}^j2$ where $j \leqslant n+1$, put

$$yM^1[H(i) = a]$$

iff $j > i$ and the following condition holds:

Whenever $\eta_1 \in X_t$, $t \supseteq y$, then

$$\eta_1 < \eta_2 \in \bigcup_{t \in {}^n2} X_t \rightarrow p_{\eta_1\eta_2} \Vdash F_{\eta_1}(i) = a$$

and

$$\xi < \eta_1, \quad \xi \in \bigcup_{t \in {}^n2} X_t \rightarrow q_{\xi\eta_1} \Vdash F_{\eta_1}(i) = a.$$

Similarly, if $0 \leqslant i \leqslant n$ and $y \in {}^j2$, $z \in {}^{j'}2$ where $j, j' \leqslant n+1$, put

$$\langle y, z\rangle M^2[K(i) = b]$$

if and only if $i < \min(j, j')$, $y < z$, and

$$\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2}\rangle \Vdash G(i) = b$$

whenever $\eta_1 \in X_{t_1}$, $\eta_2 \in X_{t_2}$, $t_1 \supseteq y$, $t_2 \supseteq z$.

From this definition, it is immediate that $\langle p_{\eta\xi}, q_{\eta\xi} \mid \eta < \xi$ in $X\rangle$ witnesses to $\langle J, X_t\rangle_{t \in {}^{n+1}2}$ satisfying $M = \langle M^1, M^2\rangle$, if indeed the latter is an $(n+1)$-sprout. It remains to be proved that M is an $(n+1)$-sprout. Condition (b) of Definition 3.2 is almost immediate from the definitions of M^1 and M^2. The "only if" part of (a) is clear from definition. The "if" part of (a) is argued differently for $i < n$ and $i = n$: if $i < n$, then, because $\langle p'_{\eta\xi}, q'_{\eta\xi} \mid \eta < \xi$ in $X'\rangle$ witnesses to $\langle J', X'_t\rangle_{t \in {}^n2}$ satisfying M' and because $X_{t\frown\langle j\rangle} \subseteq X'_t$, it follows that whenever x or $\langle x, y\rangle$ stands in the relation M'^1 or M'^2 to a sentence, it also stands in the relation M^1 or M^2 (respectively) to that sentence. So the fact that M' satisfies Definition 3.2(a) implies that M satisfies Definition 3.2(a) for $i<n$. The case $i=n$ for K is handled by applying 3.12: it is an easy consequence of 3.12(B) and 3.12(C) that

$$\langle t_i, t'_i\rangle M^2[K(n)=s^i(n)]$$

the case $i=n$ for H goes through because s_η was stabilized for $\eta \in X''_t$.

Why is Definition 3.2(d) true of M? By the definition of M^1 and M^2, it is clear that, if $x<y$ are in ${}^{n+1}2$, then, for any $\eta_1 \in X_x$, $\eta_2 \in X_y$,

$$p_{\eta_1\eta_2} \Vdash \bar{F}_{\eta_1}(n+1) = M^1[x],$$
$$q_{\eta_1\eta_2} \Vdash \bar{F}_{\eta_2}(n+1) = M^1[y]$$

and

$$\langle p_{\eta_1\eta_2}, q_{\eta_1\eta_2}\rangle \Vdash \bar{G}_{\eta_1\eta_2}(n+1) = M^2[x, y].$$

By Lemma 3.1,

$$R(\overline{M^1[x]}(i), \overline{M^1[x]}(i), \overline{M^2[x, y]}(i)),$$

as desired. Thus 3.11 follows from 3.12.

Now we show how to prove 3.12. Suppose we are given, for a fixed i between 0 and $k-1$, objects X^i_t, $p^i_{\eta\xi}$, $q^i_{\eta\xi}$ satisfying 3.12. We propose to obtain objects X^{i+1}_t, $p^{i+1}_{\eta\xi}$, $q^{i+1}_{\eta\xi}$ also satisfying 3.12, which will complete the proof. For each $\eta \in X^i_{t_{i+1}}$, $\xi \in X^i_{t_{i+1}}$ (recall that $\langle t_{i+1}, t'_{i+1}\rangle$ is the $(i+1)$st pair in our enumeration of all $\langle t, t'\rangle$ where $t<t'$ are in ${}^{(n+1)}2$), choose $p^*_{\eta\xi}$ extending $p^i_{\eta\xi}$, $q^*_{\eta\xi}$ extending $q^i_{\eta\xi}$, and $s_{\eta\xi} \in {}^{\omega}\omega$ such that

$$\langle p^*_{\eta\xi}, q^*_{\eta\xi}\rangle \Vdash \overline{G_{\eta\xi}}(n+1) = s_{\eta\xi}.$$

Let A be the function whose domain is $X^i_{t_{i+1}} \times X^i_{t_{i+1}}$ given by

$$A(\xi, \eta) = \langle s_{\eta\xi}, [p^*_{\eta\xi}]_{\nu-4i-4}, [q^*_{\eta\xi}]_{\nu-4i-4}\rangle.$$

Since there are at most $\aleph_{\nu-4i-3}$ $\sim_{\nu-4i-4}$ equivalence classes, it is easy to recast A as a map from $X^i_{t_{i+1}} \times X^i_{t'_{i+1}}$ into ${}^{\omega}\omega \times \aleph_{\nu-4i-3} \times \aleph_{\nu-4i-3}$. By induction hypotheses $X^i_{t_{i+1}}$ and $X^i_{t'_{i+1}}$ each has cardinality $\aleph_{\nu-4i}$. Now apply Lemma 3.10 (strictly speaking, first replace $X^i_{t'_{i+1}}$ by a subset of cardinality $\aleph_{\nu-4i-2}$) to obtain sets $X^{i+1}_{t_{i+1}} \subseteq X^i_{t_{i+1}}$, $X^{i+1}_{t'_{i+1}} \subseteq X^i_{t'_{i+1}}$ each of each of cardinality $\aleph_{\nu-4i-4}$, such that A, as recast, is constant on $X^{i+1}_{t_{i+1}} \times X^{i+1}_{t'_{i+1}}$. In other words, for each $\eta \in X^{i+1}_{t_{i+1}}$, $[p^*_{\eta\xi}]_{\nu-4i-4}$ is the same for all $\xi \in X^{i+1}_{t'_{i+1}}$; and, for each $\xi \in X^{i+1}_{t'_{i+1}}$, $[q^*_{\eta\xi}]_{\nu-4i-4}$ is the same for all $\eta \in X^{i+1}_{t_{i+1}}$; and $s_{\eta\xi}$ is the same for all $\eta \in X^{i+1}_{t_{i+1}}$, $\xi \in X^{i+1}_{t'_{i+1}}$. (The lemma would give us sets of cardinality $\aleph_{\nu-4i}$, $\aleph_{\nu-4i-2}$, respectively, but we have no need of such precision.) If $t \neq t_{i+1}$, $t \neq t'_{i+1}$, $t \in {}^{(n-1)}2$, set $X^{i+1}_t = X^i_t$. If $\eta \in X^{i+1}_{t_{i+1}}$, let $B(\eta)$ be the $\sim_{\nu-4i-4}$ equivalence class which contains all $p^*_{\eta\xi}$, $\xi \in X^{i+1}_{t'_{i+1}}$. Similarly, for $\xi \in X^{i+1}_{t'_{i+1}}$, let $B(\xi)$ be the $\sim_{\nu-4i-4}$ equivalence class of the $q^*_{\eta\xi}$ where $\eta \in X^{i+1}_{t_{i+1}}$. Let $\eta \in X^{i+1}_{t_{i+1}}$. Note that, for any

$$\eta' \in X^{i+1} \stackrel{\text{def}}{=} \bigcup_{t \in {}^{(n+1)}2} X^{i+1}_t,$$

$p^i_{\eta\eta'}$ (if $\eta < \eta'$) or $q^i_{\eta'\eta}$ (if $\eta' < \eta$) has an extension whose $\sim_{\nu-4i-4}$ equivalence class is $B(\eta)$. (To see this for the case $\eta < \eta'$, let ξ be a fixed member of $X^{i+1}_{t_{i+1}}$. Since $p^i_{\eta\eta'}$ and $p^i_{\eta\xi}$ are $\sim_{\nu-4i}$ equivalent, they have the same $\sim_{\nu-4i-4}$ and hence the same $\sim_{\nu-4i-4}$ equivalence classes represented among their extensions; in particular, $B(\eta)$, the $\sim_{\nu-4i-4}$ equivalence class of $p^*_{\eta\xi}$ which extends $p^i_{\eta\xi}$, must be represented among the extensions of $p^i_{\eta\eta'}$.) Let $p^{i+1}_{\eta\eta'}$ be an extension of $p^i_{\eta\eta'}$ whose $\sim_{\nu-4i-4}$ equivalence class is $B(\eta)$ whenever $\eta' > \eta$, $\eta' \in X^{i+1}$. Similarly, if $\eta' < \eta$, $\eta' \in X^{i+1}$, let $q^{i+1}_{\eta'\eta}$ be an extension of $q^i_{\eta'\eta}$ whose $\sim_{\nu-4i-4}$ equivalence class is $B(\eta)$. The same argument shows that, for any $\xi \in X^{i+1}_{t_{i+1}}$, we can, for $\xi' < \xi$, $\xi' \in X^{i+1}$, take $q^{i+1}_{\xi'\xi}$ to be an extension of $q^i_{\xi'\xi}$ whose $\sim_{\nu-4i-4}$ equivalence class is $B(\xi)$, and similarly for $p^{i+1}_{\xi\xi'}$ where $\xi' > \xi$, $\xi' \in X^{i+1}$. In all other cases not mentioned so far, take $p^{i+1}_{\eta\xi} = p^i_{\eta\xi}$, $q^{i+1}_{\eta\xi} = q^i_{\eta\xi}$. If we take s^{i+1} to be that sequence such that

$$A(\eta, \xi) = \langle s^{i+1}, , \rangle$$

for all $\eta \in X^{i+1}_{t_{i+1}}$, $\xi \in X^{i+1}_{t_{i+1}}$, then the objects X^{i+1}_t, $p^{i+1}_{\eta\xi}$, $q^{i+1}_{\eta\xi}$, s^{i+1} that we have just defined clearly satisfy the conditions of 3.12.

4.

We are still assuming the hypotheses stated at the beginning of Section 3. It is the aim of this section to obtain the functions F_η, $G_{\eta\xi}$, $\eta<\xi<\aleph_{\omega_1}$, upon whose existence the arguments of Section 3 depend, i.e., to prove 3($\#$), stated in the sixth paragraph of Section 3.

The existence of the desired functions will be obtained by transfinite recursion, i.e., we intend to show: If γ is an ordinal and $\langle F_\eta \mid \eta < \gamma\rangle, \langle G_{\eta\xi} \mid \eta < \xi < \gamma\rangle$ satisfy the conditions of ($\#$), then there is an ordinal λ_γ and there are functions F_γ, $G_{\eta\gamma}$ (for $\eta < \gamma$) such that $\langle F_\eta \mid \eta \leq \gamma\rangle, \langle G_{\eta\xi} \mid \eta < \xi \leq \gamma\rangle$ satisfy the conditions of ($\#$). Our strategy is this: first we prove this for the case where $\gamma = \omega$ and every λ_η is countable, and then we reduce the general case to this special case. At the end of the Section, we sketch my original proof of 3($\#$), which uses several metamathematical techniques.

Lemma 4.1. *If $F: {}^\omega\omega \to \mathcal{P}$ is Cohen continuous, then there is a countable set $C \subseteq {}^\omega\omega$ such that $\{\alpha \in {}^\omega\omega \mid (\exists \beta \in C)\ F(\alpha) E \beta\}$ is comeager (i.e., if F is restricted to a suitable comeager set, only countably many E equivalence classes occur in the range).*

Proof. Suppose $p \in {}^{<\omega}\omega$, say $p \in {}^n\omega$. If $\alpha \in {}^\omega\omega$, let α_p be the result of replacing the first n entries of α by p, i.e.

$$\alpha_p = \langle p(0), p(1), \ldots, p(n-1), \alpha(n), \alpha(n+1), \ldots\rangle.$$

Let

$$K = \{\langle \alpha, \beta\rangle \mid \alpha, \beta \in {}^\omega\omega \text{ and } (\forall p \in {}^{<\omega}\omega)(\exists q \in {}^{<\omega}\omega)$$
$$(F(\alpha_p) E F(\beta_q)) \text{ and } (\forall q \in {}^{<\omega}\omega)(\exists p \in {}^{<\omega}\omega)(F(\alpha_p) E F(\beta_q))\}.$$

Thus $\langle\alpha,\beta\rangle\in K$ iff the same E equivalence classes are represented among the $F(\alpha_p)$ as among the $F(\beta_q)$. The set K has the property of Baire—this may be seen by noting that it has a $\mathbf{\Pi}^1_1$ definition and is accordingly coanalytic. The conclusion of Lemma 4.1 will easily follow if we can show that K is comeager in ${}^\omega\omega\times{}^\omega\omega$. For then let $W=\{\alpha\in{}^\omega\omega\mid\{\beta\mid\langle\alpha,\beta\rangle\in K\}$ is comeager in ${}^\omega\omega\}$. By the Kuratowski–Ulam theorem, which is the analogue of Fubini's theorem for category theory (see Oxtoby [9]), W is comeager in ${}^\omega\omega$. We claim that $W\times W\subseteq K$ (which proves Lemma 4.1 if we take $C=\{F(\alpha_p)\mid p\in{}^{<\omega}\omega,\ F(\alpha_p)\in{}^\omega\omega\}$ where α is a fixed member of W). If $\alpha_1,\alpha_2\in W$, then $\{\beta\mid\langle\alpha_1,\beta\rangle\in K\}$ and $\{\beta\mid\langle\alpha_2,\beta\rangle\in K\}$, being comeager, contain a common element β. But K is an equivalence relation, so $\langle\alpha_1,\beta\rangle\in K$, $\langle\alpha_2,\beta\rangle\in K$ imply $\langle\alpha_1,\alpha_2\rangle\in K$. Thus $W\times W\subseteq K$. (If the proof had been written in terms of forcing, we would have taken W to be an appropriate collection of "generic" objects, and then chosen β to be "generic" with respect to both α_1 and α_2.)

Why is K comeager? First note that K is the intersection of countably many sets, K_p and K'_p (for $p\in{}^{<\omega}\omega$) where

$$K_p=\{\langle\alpha,\beta\rangle\mid\alpha,\beta\in{}^\omega\omega\wedge(\exists q\in{}^{<\omega}\omega)(F(\alpha_p)EF(\beta_q))$$

and

$$K'_p=\{\langle\alpha,\beta\mid\alpha,\beta\in{}^\omega\omega\wedge(\exists q\in{}^{<\omega}\omega)(F(\alpha_q)EF(\beta_p)).$$

It suffices to see that all the K_p and K'_p are comeager. By symmetry, it is enough to consider K_p. Since K_p has a $\mathbf{\Pi}^1_1$ definition, it is coanalytic, so both it and its complement have the property of Baire. To say that ${}^\omega\omega-K_p$ has the property of Baire means that there is an open set U and a comeager set S such that

$$[({}^\omega\omega\times{}^\omega\omega)-K_p]\cap S=U\cap S.$$

Assume by way of contradiction that K_p isn't comeager. Thus $U\neq\emptyset$; say U includes the basic open neighborhood $[r,s]=\{\langle\alpha,\beta\rangle\in{}^\omega\omega\times{}^\omega\omega\mid\alpha\supseteq r,\ \beta\supseteq s\}$, where $r,s\in{}^n\omega$ and n is at least as great as the length of p. Let

$$u=\langle p(0),\ldots,p(m-1),r(m),\ldots,r(n-1)\rangle$$

where $m=$ length p. Since there is a set of $\langle\alpha,\beta\rangle$ comeager relative to $[r,s]$ such that $F(\alpha_p)EF(\beta_q)$ fails for all q (a restatement of: "complement of K_p is comeager relative to $[r,s]$"), there must be a set T' of $\langle\alpha,\beta\rangle$ comeager relative to $[u,s]$ such that $F(\alpha)EF(\beta_q)$ fails for all q. The set $\{\langle\alpha,\beta\rangle\mid\langle\alpha,\beta\rangle\in T'\}$, which we denote T, is the translation of T' from $[u,s]$ to $[u,u]$, so T is comeager relative to $[u,u]$. Moreover, taking $q=u$ in the defining property of T', one has: $\langle\alpha,\beta\rangle\in T\rightarrow F(\alpha)EF(\beta)$ fails.

Now, following a suggestion of the referee, we can finish the argument in the following simple way. In the proof of Corollary 2.4, it was shown that any comeager subset of ${}^\omega2\times{}^\omega2$ includes a set of the form $(P\times P)-\{\langle\alpha,\alpha\rangle\mid\alpha\in P\}$

where P is nonempty and perfect. Precisely the same argument yields this conclusion for comeager subsets of ${}^{\omega}\omega \times {}^{\omega}\omega$, and, indeed, of $[u, u] \cap ({}^{\omega}\omega \times {}^{\omega}\omega)$.

Let $T^* = T \cap [u, u] \cap \{\langle\alpha, \beta\rangle \mid F(\alpha) \in {}^{\omega}\omega, F(\beta) \in {}^{\omega}\omega\}$. T^* is comeager relative to $[u, u]$ because T is comeager relative to $[u, u]$ and the third set is comeager, being the product of $\{\alpha \mid F(\alpha) \in {}^{\omega}\omega\}$ with itself. By the preceding paragraph, there is a perfect subset $P \subseteq {}^{\omega}\omega \cap [u]$ such that

$$(P \times P) - \{\langle\alpha, \alpha\rangle \mid \alpha \in P\} \subseteq T^*.$$

Since F is $1-1$ on P (in fact: $\alpha, \beta \in P$, $\alpha \neq \beta \rightarrow \langle\alpha, \beta\rangle \in T \rightarrow F(\alpha) FF(\beta)$ fails), $\{F(\alpha) \mid \alpha \in P\}$ is an uncountable analytic set (being the image of P under the continuous function $F \restriction P$). Moreover, distinct members of $\{F(\alpha) \mid \alpha \in P\}$ are E inequivalent. Let S be a perfect subset of $\{F(\alpha) \mid \alpha \in P\}$ (such exists because any uncountable analytic set includes a perfect set). S is a perfect set of mutually E inequivalent elements, contrary to the hypothesis on E.

By Lemma 2.3, there is a Cohen continuous function $J : [u, u] \rightarrow \mathcal{P}$ and a comeager subset T^* of $[u, u]$, $T^* \subseteq T$, such that: whenever $\langle\alpha, \beta\rangle \in T^*$, then $S(F(\alpha), F(\beta), J(\alpha, \beta))$. (For the definition of S, see the beginning of Section 3.) To obtain the desired contradiction, we intend to show that, contrary to hypothesis, there is a perfect set of mutually E inequivalent elements.

Corollary 4.2. *Suppose $F_i : {}^{\omega}\omega \rightarrow \mathcal{P}$ are Cohen continuous for $i \in \omega$, then there are Cohen continuous functions $F : {}^{\omega}\omega \rightarrow \mathcal{P}$, $G_i : {}^{\omega}\omega \times {}^{\omega}\omega \rightarrow \mathcal{P}$ such that:*

$$\beta, \alpha \in {}^{\omega}\omega \quad \textit{and} \quad F_i(\beta), F(\alpha), G_i(\beta, \alpha) \ \textit{total}$$
$$\rightarrow S(F_i(\beta), F(\alpha), G_i(\beta, \alpha)).$$

Proof. Apply Lemma 4.1 to obtain comeager sets $T_i \subseteq {}^{\omega}\omega$ and countable sets $C_i \in {}^{\omega}\omega$, such that:

$$\beta \in T_i \rightarrow (\exists \alpha \in C_i)\ F_i(\beta) E \alpha.$$

Since E has uncountably many equivalence classes, there is an element $\delta \in {}^{\omega}\omega$ not E equivalent to anything in $\bigcup_{i \in \omega} C_i$. Hence $F_i(\beta) E \delta$ fails for all $\beta \in T_i$. Apply Lemma 2.3 where

$$C = \{\langle\beta, \gamma\rangle \mid S(F_i(\beta), \delta, \gamma)\},$$

to obtain Cohen continuous functions

$$J_i : {}^{\omega}\omega \rightarrow \mathcal{P}$$

such that $S(F_i(\beta), \delta, J_i(\beta))$ holds whenever $J_i(\beta) \in {}^{\omega}\omega$. Put $F(\alpha) = \delta$ for all $\alpha \in {}^{\omega}\omega$, and $G_i(\beta, \alpha) = J_i(\beta)$ for all β, $\alpha \in {}^{\omega}\omega$. It is clear that G_i and F satisfy the conclusion of Corollary 4.2.

Lemma 4.3. *Suppose $\langle F_\eta \mid \eta < \tau\rangle$ is such that, for each $\eta < \tau$, $F_\eta : {}^{\omega}(\lambda_\eta) \rightarrow \mathcal{P}$ is Cohen continuous. If λ is an infinite cardinal such that $\lambda \geqslant \operatorname{card} \tau$ and $\lambda \geqslant \operatorname{card} \lambda_\eta$,* *each η, then there are Cohen continuous functions*

$$F : {}^{\omega}\lambda \to \mathcal{P}, \qquad G_\eta : {}^{\omega}\lambda_\eta \times {}^{\omega}\lambda \to \mathcal{P} \qquad (\eta < \tau)$$

such that, whenever $\gamma \in {}^{\omega}\lambda_\eta$, $\delta \in {}^{\omega}\lambda$ are such that $G_\eta(\gamma, \delta)$ is total, then

$$S(F_\eta(\gamma), F(\delta), G_\eta(\gamma, \delta)).$$

Proof. For each $\eta < \tau$, let $f_\eta : {}^{\omega}\lambda_\eta \to {}^{\omega}\omega$ be given by:

if $s \in {}^{n}\lambda_\eta$, $f_\eta(s) =$ the longest $t \in {}^{\omega}\omega$ of length $\leqslant n$ such that $s \Vdash t \subseteq F_\eta(\alpha)$.

Let Q be a map of λ onto $\bigcup_{\eta<\tau}(\{\eta\} \times {}^{\omega}\lambda_\eta)$. If $\sigma < \lambda$, let $Q(\sigma) = \langle \eta_\sigma, s_\sigma \rangle$. We propose to define functions $g_\eta : {}^{\omega}\lambda_\eta \times {}^{\omega}\lambda \to {}^{\omega}\omega$ $(\eta < \tau)$ and $f : {}^{\omega}\lambda \to {}^{\omega}\omega$ having the following properties:

(1) $s \subseteq s' \wedge t \subseteq t' \to f(t) \subseteq f(t) \wedge g_\eta(s, t) \subseteq g_\eta(s', t')$. Also: If s and s' don't clash, and t and t' don't clash, then $g_\eta(s, t)$ and $g_\eta(s', t')$ don't clash.

(2) If m is a number less than each of these three numbers, length $g_\eta(s, t)$, length $f_\eta(s)$, length $f(t)$, then

$$R(\overline{f_\eta(s)}(m), \overline{f(t)}(m), \bar{g}_\eta\overline{(s, t)}(m)).$$

(3) If $t \in {}^{n+1}\lambda$ where $Q(t(n)) = \langle \eta, s \rangle$, then, for some $s' \supseteq s$, $s' \in {}^{\omega}\lambda_\eta$, one has: $g_\eta(s', t)$ has length $\geqslant n + 1$.

(4) length $f(t) \geqslant$ length t.

It is clear that such functions f and g_η (for $\eta < \tau$) exist if and only if player II has a winning strategy in the following infinite game. Player I keeps playing ordinals $\sigma_0, \sigma_1, \sigma_2, \ldots < \lambda$. Suppose we are at step n of the game, I having played $\sigma_0, \ldots \sigma_n$ up to this point. Put $\eta = \eta_{\sigma_n}$, $s = s_{\sigma_n}$, $t = \langle \sigma_0, \ldots, \sigma_n \rangle$. To avoid losing, II must declare what $f(t)$ is, he must stipulate some $s' \supseteq s$, $s' \in {}^{\omega}\lambda_\eta$, and then declare what $g_\eta(s', t)$ is. Moreover, this must be done so that (4) and the last part of (3) are satisfied, and so that conditions (1) and (2) are satisfied for all of the declarations of values II has made up through this point where s and s' range over all sequences [i.e., they are not just the specific s and s' associated with the nth move]. Player II wins a play of this game iff his every move satisfies the conditions just described. If II has a winning strategy S, then define:

$f(t) =$ that value which S declares $f(t)$ to be in response to the sequence of moves t by I.

$g_\eta(s, t) = \bigcup$(values for $g_\eta(s', t')$, $s' \subseteq s$, $t' \subseteq t$, declared by S in response to initial segments t' of t).

It is clear that the resulting f and g_η's satisfy conditions (1)–(4) above.

Therefore, to prove Lemma 4.3, it suffices to show that II has a winning strategy in the game. For, if we have functions satisfying (1)–(4), then

$$F(\delta) = \bigcup_{n \in \omega} f(\bar{\delta}(n))$$

and

$$G_\eta(\gamma, \delta) = \bigcup_{n \in \omega} g_\eta(\bar{\gamma}(n), \bar{\delta}(n))$$

satisfy the conclusion of Lemma 4.3.

However, this game is an open game from the point of view of player I (because I wins a play iff II "makes a mistake" at some point, so, if I wins a play, he knows it at some finite point, which is the defining characteristic of an open game). Hence, either player I or player II, has a winning strategy in this game (see [8]). [Roughly speaking, the standard argument is this: If I doesn't have a winning strategy, II can win by avoiding positions from which I has a winning strategy.] To show that II has a winning strategy, it therefore suffices to show that I doesn't have a winning strategy. We now proceed to do so, using Lemma 4.2.

Suppose, by way of contradiction, that S is a winning strategy for I in this game. We claim that there exist countable sets $U \subseteq \tau$ and $X_\eta \subseteq \lambda_\eta$ for $\eta \in U$ and $X \subseteq \lambda$ such that:

(a) For $\eta \in U$, $F_\eta \restriction {}^\omega X_\eta$ is Cohen continuous.

(b) $\sigma \in X$ iff $\sigma \in \lambda$, $\eta_\sigma \in U$, and $s_\sigma \in {}^\omega X_{\eta_\sigma}$.

(c) If $\langle z_0, \ldots, z_{n-1}\rangle$ is a sequence of moves on the part of II which mention only sequences in $X_\eta (\eta \in U)$ [i.e. if z_i mentions s' in declaring $g_\eta(s', t) =$ something, then $\eta \in U$ and $s' \in {}^\omega X_\eta$], then S responds with a number of X.

[We can obtain these as unions of countable sequences:

$$U = \bigcup_{m \in \omega} U^m, \quad X_\eta = \bigcup_{m \in \omega} X_\eta^m, \quad X = \bigcup_{m \in \omega} X^m,$$

where X^{m+1} contains all responses given by S to moves involving elements of U^m and the X_η^m's ($\eta \in U$), and X^{m+1} contains every σ such that $\eta_\sigma \in U^m$, $s_\sigma \in {}^\omega X_{\eta_\sigma}^m$; and, whenever $\sigma \in X^m$, then $\eta_\sigma \in U^{m+1}$, $s_\sigma \in {}^\omega X^{m+1}$; and the sequence $\langle X_\eta^m \mid m \in \omega\rangle$ is arranged to insure Cohen continuity of $F_\eta \restriction {}^\omega X_\eta$.]

Now we exhibit a play of the game in which strategy S loses. By Corollary 4.2, there are Cohen continuous functions,

$$G'_\eta : {}^\omega X_\eta \times {}^\omega \omega \longrightarrow \mathscr{P}, \quad F' : {}^\omega \omega \longrightarrow \mathscr{P}$$

satisfying the obvious modification of the conclusion of Corollary 4.2. As usual, we may find

$$g'_\eta : {}^{<\omega} X_\eta \times {}^{<\omega}\omega \longrightarrow {}^{<\omega}\omega, \quad f' : {}^{<\omega}\omega \longrightarrow \mathscr{P}$$

such that

$$G'_\eta(\gamma, \delta) = \bigcup g'_\eta(\bar{\gamma}(n), \bar{\delta}(n)), \quad F'(\delta) = \bigcup f'(\bar{\delta}(n))$$

(provided that we modify G'_η, F' slightly). To defeat S, proceed as follows. Suppose σ_0 is the initial move dictated by S. Choose $u_0 \in {}^{<\omega}\omega$, $s'_0 \in {}^{\omega}X_{\eta_{\sigma_0}}$ such that $s'_0 \supseteq s_0$ and $f'(u_0)$, $g'_\eta(s'_0, u_0)$ have lengths at least 1. Have II declare:

$$f(\langle\sigma_0\rangle) = f'(u_0), \quad g_\eta(s'_0, \langle\sigma_0\rangle) = g'_\eta(s'_0, u_0).$$

Suppose play continues in this way with I playing $\sigma_0, \ldots, \sigma_{n-1}$ and II choosing $u_0 \subseteq u_1 \subseteq \cdots \subseteq u_{n-1}$ in ${}^{\omega}\omega$. If I now plays $\sigma_n = \sigma$ let $u_n \in {}^{\omega}\omega$ extend u_{n-1} and $s'_n \in {}^{\omega}X_{\eta_\sigma}$, $s'_n \supseteq s_\sigma$ be such that $f'(u_n)$, $g'_\eta(s'_n, u_n)$ have lengths $\geqslant n+1$. Have II declare that $f(\langle \sigma_0, \ldots, \sigma_n \rangle) = f'(u_n)$ and $g_\eta(s'_n, \langle \sigma_0, \ldots, \sigma_n \rangle) = g'_\eta(s'_n, u_n)$.

This play of the game wins for II because condition (2) follows from the fact that G'_η and F' satisfy the conclusion of Corollary 4.2. Condition (4) and the last part of (3) for s'_n were specifically provided for in the choice of u_n and s'_n. The consistency condition (1) is evident.

Clearly Lemma 4.3 enables one to construct the functions demanded by (#).

We now indicate briefly how to use metamathematical methods to obtain a shorter proof of 3(#). For ease of exposition, we assume that E is a pure Π^1_1 equivalence relation (i.e. Π^1_1 without parameters). Thus, we may assume that, in the representation,

$$\neg(\alpha E \beta) \quad \text{iff } (\exists \gamma \in {}^{\omega}\omega)(\forall n \in \omega)\, R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)),$$

R is recursive. (To modify the following argument to handle Π^1_1's simply assume that all the models of set theory used contain the defining parameters.) It should be added that it is possible to modify our argument here to make use of only ω_1 infinite cardinals.

Let σ_0 and σ_1 be theorems of ZFC such that every Cohen extension of a model of σ_0 is a model of σ_1, σ_1 is strong enough to make the proof of I work, and σ_0 is strong enough to imply that there are at least $\aleph_{\omega_1+1}$ infinite cardinals and to make arguments in the sequel work.

I. *If M is a transitive model of σ_1, this $M \vDash E \cap M$ is an equiv. relation having uncountably many equiv. classes and, for all $\alpha, \beta \in M \cap {}^{\omega}\omega$,*

$$\neg(\alpha E \beta) \leftrightarrow M \vDash (\exists \gamma \in {}^{\omega}\omega)(\forall n \in \omega)\, R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n)).$$

Proof. The second claim holds because Σ^1_1 statements are absolute with respect to M, so $(\exists \gamma \in {}^{\omega}\omega)(\exists n \in \omega)\, R(\bar{\alpha}(n), \bar{\beta}(n), \bar{\gamma}(n))$ holds in the real world if and only if it holds in the sense of M. Thus $E \cap M$ has the same Π^1_1 definition in M as E has in the real world. If the first claim fails, then M contains a member $\delta \in {}^{\omega}\omega$, which lists representatives for all E equivalence classes in M, i.e. every $\alpha \in M \cap {}^{\omega}\omega$ is E equivalent to some $(\delta)_n$ where $(\delta)_n(m) = \delta(2^n 3^m)$. But, since $(\forall \alpha \in {}^{\omega}\omega)(\exists n)(\alpha E (\delta)_n)$ is a Π^1_1 sentence true in M (if we replace E by its Π^1_1 definition in M), it is true in the real world, by the Mostowski absoluteness theorem, which contradicts our assumptions that E has uncountably many equivalence classes.

Now, using the reflection theorem (starting from *I.*), we get a countable transitive model M of $\sigma_0 + V = L$ such that, if there is no sequence of functions satisfying (#), then the same is true in M. For each $\alpha < \aleph_1^M$, let $\mathbb{P}_\alpha$ be the partial ordering for mapping ω onto $\omega^M_{\alpha+1}$, i.e. $\mathbb{P}_\alpha$ consists of functions

$$p : n \to \omega^M_{\alpha+1}$$

as n ranges over ω.

II. *If $M \models \mathbb{P}$ is a partial ordering, and G and H are $\mathbb{P}$ generic/M, then the same E equivalence classes are realized in $M[G]$ as in $M[H]$.*

Remarks concerning proof. The proof resembles that of Lemma 4.1 (and, indeed, they are closely related propositions). By choosing a K which is $\mathbb{P}$ generic over both $M[G]$ and $M[H]$, one may in effect assume that $G \times H$ is $\mathbb{P} \times \mathbb{P}$ generic. One then shows, if the conclusion of II fails, that it is possible to construct an infinite binary tree through $\mathbb{P}$ whose paths produce a perfect set of mutually E inequivalent elements, contrary to assumption.

III. *For each $\eta < \aleph^M_{\omega_1}$ there is a term t_η in the forcing language for $\mathbb{P}_\eta$ such that, whenever $f_\eta : \omega \to \omega^M_{\eta+1}$ is $\mathbb{P}_\eta$ generic and, for some $\xi < \eta$, $f_\xi : \omega \to \omega^M_{\xi+1}$ is $\mathbb{P}_\xi$ generic, then*

$$t_\eta(f_\eta)^{M[f_\eta]}$$

is not E equivalent to any member of $M[f_\xi]$.

Outline of Proof. For each $\xi < \eta$, let $f^\xi_\eta(m) = f_\xi(m)$ if $f_\eta(m) < \omega^M_{\xi+1}$, otherwise 0. It is easy to check that f^ξ_η is $\mathbb{P}_\xi$ generic/M. Moreover, for each $\xi < \eta$,

$$M[f_\eta] \models {}^\omega\omega \cap M[f^\xi_\eta] \text{ is countable}$$

(because $\omega^M_{\xi+1}$ is countable in the sense of $M[f_\eta]$). Hence

$$M[f_\eta] \models \bigcup_{\xi<\eta} ({}^\omega\omega \cap M[f_\eta]) \quad \text{is countable.}$$

Clearly there is a term t_η in the $\mathbb{P}_\eta$ language such that $t_\eta(f_\eta)^{M[f_\eta]}$ is always the first member of $M[f_\eta]$ (in the sense of the canonical ordering of $L[f_\eta]$) not E equivalent to any member of $\bigcup_{\xi<\eta}({}^\omega\omega \cap M[f_\eta])$. (This also uses I above.) Now apply II to see that t_η has the property claimed in III.

One may assume that the sequence $\langle t_\eta \mid \eta < \aleph^M_{\omega_1}\rangle$ is constructed in M since the condition which the t_η's must satisfy can be formulated by means of forcing. Similarly, one can find within M a sequence $\langle \sigma_{\eta\xi} \mid \eta < \xi < \aleph^M_{\omega_1}\rangle$ such that

$$\langle 0, 0\rangle \Vdash (\forall n \in \omega)\, R(\bar{t}_\eta(n), \bar{t}_\xi(n), \bar{\sigma}_{\eta\xi}(n)).$$

Working within M, one obtains a sequence satisfying ($\#$) of Section 3 as follows: Let $\lambda_\eta = \omega^M_{\eta+1}$, and put, for $f \in {}^\omega\lambda_\eta$, $g \in {}^\omega\lambda_\xi$

$$F_\eta(f)(m) = n \quad \text{iff } (\exists p \subseteq f)(p \Vdash t_\eta(m) = n)$$

and

$$G_{\eta\xi}(f, g)(m) = n \quad \text{iff } (\exists p \subseteq f, q \subseteq g)(\langle p, q \rangle \Vdash \sigma_{\eta\xi}(m) = n).$$

Since M thinks that there is such a sequence, the world also thinks so, because M was chosen so as to agree with the real world on this point.

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